Mechanics of Continuous Media in $(\overline{L}_n, g)$-Spaces.
II. Relative velocity and deformations

S. Manoff
Bulgarian Academy of Sciences,
Institute for Nuclear Research and Nuclear Energy,
Blvd. Tzarigradsko Chaussee 72,
1784 Sofia - Bulgaria

E-mail address: smanov@inrne.bas.bg

Abstract

Basic notions of continuous media mechanics are introduced for spaces with affine connections and metrics. The physical interpretation of the notion of relative velocity is discussed. The notions of deformation velocity tensor, shear velocity, rotation (vortex) velocity, and expansion velocity are introduced. Different types of flows are considered.

PACS numbers: 11.10.-z; 11.10.Ef; 7.10.+g; 47.75.+f; 47.90.+a; 83.10.Bb

Contents

1 Introduction

The notion of relative velocity is closely related to the notions of deformation velocity tensor and its kinematic characteristics (shear, rotation, and expansion). On the other side, the friction in a continuous media could be described in analogous way as the deformation. This give rise to considerations of friction "velocity" tensor and its kinematic characteristics.

In Section 1 the introduction and the physical interpretation of the notion of relative velocity is discussed. The notions of deformation velocity tensor, shear velocity, rotation (vortex) velocity and vortex vector as well as expansion velocity are introduced for $(\overline{L}_n, g)$-spaces. In Section 2 the notions of friction velocity and it kinematic characteristics is introduced and considered. In Section 3 different types of flows are considered.

All considerations are given in details (even in full details) for those readers who are not familiar with the considered problems.

Remark. The present paper is the second part of a larger research report on the subject with the title "Contribution to continuous media mechanics in $(\overline{L}_n, g)$-spaces" and with the following contents:

I. Introduction and mathematical tools.

II. Relative velocity and deformations.

III. Relative accelerations.

IV. Stress (tension) tensor.

The parts are logically self-dependent considerations of the main topics considered in the report.

2 Relative velocity. Deformation velocity, shear velocity, rotation (vortex) velocity, and expansion velocity

The notion *relative velocity* vector field (relative velocity) ${}_{rel}v$ can be defined (regardless of its physical interpretation) as the orthogonal to a non-isotropic vector field u projection of the first covariant derivative (along the same non-isotropic vector field u) of (another) vector field ξ, i.e.

$$\begin{gathered} {}_{rel}v = \overline{g}(h_u(\nabla_u \xi)) = g^{ij} \cdot h_{\overline{j}\overline{k}} \cdot \xi^k{}_{;l} \cdot u^l \cdot e_i = \\ = g^{ij} \cdot f^m{}_j \cdot f^n{}_k \cdot h_{mn} \cdot \xi^k{}_{;l} \cdot u^l \cdot e_i \ , \\ e_i = \partial_i \text{ (in a co-ordinate basis)}, \end{gathered} \tag{1}$$

where (the indices in a co-ordinate and in a non-co-ordinate basis are written in both cases as Latin indices instead of Latin and Greek indices)

$$h_u = g - \frac{1}{e} \cdot g(u) \otimes g(u) \ , \ h_u = h_{ij} \cdot e^i.e^j \ , \overline{g} = g^{ij} \cdot e_i.e_j, \tag{2}$$

$$\begin{gathered} \nabla_u \xi = \xi^i{}_{;j} \cdot u^j \cdot e_i \\ \xi^i{}_{;j} = e_j \xi^i + \Gamma^i_{kj} \cdot \xi^k \ , \ \Gamma^i_{kj} \neq \Gamma^i_{jk} \ , \end{gathered} \tag{3}$$

$$\begin{gathered} g = g_{ij} \cdot e^i.e^j, \qquad g_{ij} = g_{ji} \ , \\ e^i.e^j = \frac{1}{2} \cdot (e^i \otimes e^j + e^j \otimes e^i) \ , \end{gathered} \tag{4}$$

$$
\begin{gathered}
e = g(u,u) = g_{\overline{ij}} \cdot u^i \cdot u^j = u_{\overline{i}} \cdot u^i \neq 0 \\
g(u) = g_{i\overline{k}} \cdot u^k = u_i = g_{ik} \cdot u^{\overline{k}} \ , \ u^{\overline{k}} = f^k{}_l \cdot u^l \ , \\
e_i . e_j = \frac{1}{2} \cdot (e_i \otimes e_j + e_j \otimes e_i) \ ,
\end{gathered} \tag{5}
$$

$$
\begin{gathered}
h_u(\nabla_u \xi) = h_{i\overline{j}} \cdot \xi^j{}_{;k} \cdot u^k \cdot e^i \\
h_{ij} = g_{ij} - \frac{1}{e} \cdot u_i \cdot u_j \ .
\end{gathered} \tag{6}
$$

In a co-ordinate basis

$$
\begin{gathered}
e_j \xi^i = \xi^i{}_{,j} = \partial_j \xi^i = \partial \xi^i / \partial x^j , \\
e^j = dx^j , \qquad e_i = \partial_i = \partial / \partial x^i , \qquad u = u^i \cdot \partial_i ,
\end{gathered}
$$

Every contravariant vector field ξ can be written by means of its projection along and orthogonal to u in two parts - one collinear to u and one - orthogonal to u, i.e.

$$
\xi = \frac{l}{e} \cdot u + h^u[g(\xi)] \ = \frac{l}{e} \cdot u + \overline{g}[h_u(\xi)] \ , \tag{7}
$$

where

$$
\begin{gathered}
l = g(\xi, u) \ , \qquad h^u = \overline{g} - \frac{1}{e} \cdot u \otimes u \ , \\
\xi = \xi^i \cdot \partial_i = \xi^k \cdot e_k \ , \qquad h^u = h^{ij} \cdot e_i . e_j \ ,
\end{gathered} \tag{8}
$$

$$
\overline{g}(h_u)\overline{g} = h^u \ , \quad h_u(\overline{g})(g) = h_u \ , \quad h^u(g)(\overline{g}) = h^u \ , \quad g(h^u)g = h_u \ . \tag{9}
$$

Therefore, $\nabla_u \xi$ can be written in the form

$$
\nabla_u \xi = \frac{\overline{l}}{e} \cdot u + \overline{g}[h_u(\nabla_u \xi)] = \frac{\overline{l}}{e} \cdot u + {}_{rel}v \ , \quad \overline{l} = g(\nabla_u \xi, u) \tag{10}
$$

and the connection between $\nabla_u \xi$ and ${}_{rel}v$ is obvious. Using the relation [1] between the Lie derivative $\pounds_\xi u$ and the covariant derivative $\nabla_\xi u$

$$
\begin{gathered}
\pounds_\xi u = \nabla_\xi u - \nabla_u \xi - T(\xi, u) \\
T(\xi, u) = T_{ij}{}^k \cdot \xi^i \cdot u^j \cdot e_k \ ,
\end{gathered} \tag{11}
$$

$$
T_{ij}{}^k = -T_{ji}{}^k = \Gamma^k_{ji} - \Gamma^k_{ij} - C_{ij}{}^k \quad \text{(in a non-co-ordinate basis } \{e_k\}) \ ,
$$

$$
[e_i, e_j] = \pounds_{e_i} e_j \ = C_{ij}{}^k \cdot e_k \ ,
$$

$$
T_{ij}{}^k = \Gamma^k_{ji} - \Gamma^k_{ij} \quad \text{(in a co-ordinate basis } \{\partial_k\} \text{)} \ ,
$$

one can write $\nabla_u \xi$ in the form

$$
\nabla_u \xi = (k)g(\xi) - \pounds_\xi u \ = k[g(\xi)] - \pounds_\xi u , \tag{12}
$$

or, taking into account the above expression for ξ, in the form

$$
\nabla_u \xi = k[h_u(\xi)] + \frac{l}{e} \cdot a - \pounds_\xi u \ ,
$$

where

$$
\begin{gathered}
k[g(\xi)] = \nabla_\xi u - T(\xi, u) \\
k = (u^i{}_{;l} - T_{lk}{}^i \cdot u^k) \cdot g^{lj} \cdot e_i \otimes e_j \ ,
\end{gathered} \tag{13}
$$

$$
\begin{gathered}
k[g(u)] = k(g)u = k^{ij} \cdot g_{\overline{jk}} \cdot u^k \cdot e_i = \\
= a = \nabla_u u = u^i{}_{;j} \cdot u^j \cdot e_i \ .
\end{gathered} \tag{14}
$$

For $h_u(\nabla_u \xi)$ it follows that

$$
h_u(\nabla_u \xi) = h_u(\frac{l}{e} \cdot a - \pounds_\xi u) + h_u(k) h_u(\xi) \ , \tag{15}
$$

where

$$h_u(k)h_u(\xi) = h_{i\overline{k}} \cdot k^{kl} \cdot h_{\overline{l}j} \cdot \xi^j \cdot e^i ,$$
$$h_u(u) = 0, \qquad u(h_u) = 0,$$
$$h_u(k)h_u(u) = 0, \qquad (u)h_u(k)h_u = 0.$$

If we introduce the abbreviation

$$d = h_u(k)h_u = h_{i\overline{k}} \cdot k^{kl} \cdot h_{\overline{l}j} \cdot e^i \otimes e^j = d_{ij} \cdot e^i \otimes e^j \ , \tag{16}$$

the expression for ${}_{rel}v$ can take the form

$${}_{rel}v = \overline{g}[h_u(\nabla_u \xi)] = \overline{g}(h_u)(\frac{l}{e} \cdot a - \pounds_\xi u) + \overline{g}[d(\xi)] =$$

$$= [g^{ik} \cdot h_{\overline{kl}} \cdot (\frac{l}{e} \cdot a^l - \pounds_\xi u^l) + g^{ik} \cdot d_{\overline{kl}} \cdot \xi^l] \cdot e_i \ = \ {}_{rel}v^i \cdot e_i \ , \tag{17}$$

or

$$g({}_{rel}v) = h_u(\nabla_u \xi) = h_u(\frac{l}{e} \cdot a - \pounds_\xi u) + d(\xi) \ . \tag{18}$$

For the special case when the vector field ξ is orthogonal to u, i.e. $\xi = \overline{g}[h_u(\xi)]$, and the Lie derivative of u along ξ is zero, i.e. $\pounds_\xi u = 0$, then the relative velocity can be written in the form

$$g({}_{rel}v) = d(\xi) \tag{19}$$

or in the form

$${}_{rel}v = \overline{g}[d(\xi)].$$

Remark. All further calculations leading to a useful representation of d are quite straightforward. The problem here was the finding out a representation of $h_u(\nabla_u \xi)$ in the form (15) which is not a trivial task.

2.1 Deformation velocity, shear velocity, rotation (vortex) velocity, and expansion velocity

The covariant tensor field d is a generalization for $(\overline{L}_n, g)$-spaces of the well known *deformation velocity* tensor for V_n-spaces [2], [3] . It is usually represented by means of its three parts: the trace-free symmetric part, called *shear velocity* tensor (shear), the anti-symmetric part, called *rotation velocity* tensor (rotation) and the trace part, in which the trace is called *expansion velocity* (expansion) invariant.

After some more complicated as for V_n-spaces calculations, the deformation velocity tensor d can be given in the form

$$\begin{aligned} d = h_u(k)h_u = h_u(k_s)h_u + h_u(k_a)h_u = \\ = \sigma + \omega + \tfrac{1}{n-1} \cdot \theta \cdot h_u \ . \end{aligned} \tag{20}$$

The symmetric trace-free tensor σ is the *shear velocity* tensor (shear),

$$\begin{aligned} \sigma = {}_sE - {}_sP = E - P - \tfrac{1}{n-1} \cdot \overline{g}[E-P] \cdot h_u = \sigma_{ij} \cdot e^i . e^j = \\ = E - P - \tfrac{1}{n-1} \cdot (\theta_o - \theta_1) \cdot h_u \ , \end{aligned} \tag{21}$$

$$\begin{aligned} {}_sE = E - \tfrac{1}{n-1} \cdot \overline{g}[E] \cdot h_u \ , \\ \overline{g}[E] = g^{ij} \cdot E_{\overline{ij}} = g^{\overline{ij}} \cdot E_{ij} = \theta_o \ , \end{aligned} \tag{22}$$

$$\begin{aligned} E = h_u(\varepsilon)h_u \ , \quad k_s = \varepsilon - m \ , \\ \varepsilon = \tfrac{1}{2} \cdot (u^i{}_{;l} \cdot g^{lj} + u^j{}_{;l} \cdot g^{li}) \cdot e_i . e_j \ , \end{aligned} \tag{23}$$

$$m = \frac{1}{2} \cdot (T_{lk}{}^i \cdot u^k \cdot g^{lj} + T_{lk}{}^j \cdot u^k \cdot g^{li}) \cdot e_i . e_j \ . \tag{24}$$

The symmetric trace-free tensor ${}_sE$ is the *torsion-free shear velocity* tensor, the symmetric trace-free tensor ${}_sP$ is the *shear velocity* tensor *induced by the torsion*,

$$
\begin{gathered}
{}_sP = P - \tfrac{1}{n-1} \cdot \overline{g}[P] \cdot h_u \ , \\
\overline{g}[P] = g^{kl} \cdot P_{\overline{kl}} \ = g^{\overline{kl}} \cdot P_{kl} = \theta_1 ,
\end{gathered} \tag{25}
$$

$$
\begin{gathered}
P = h_u(m)h_u \ , \qquad\qquad \theta_1 = T_{kl}{}^{k} \cdot u^l \ , \\
\theta_o = u^n{}_{;n} - \tfrac{1}{2e} \cdot (e_{,k} \cdot u^k - g_{kl;m} \cdot u^m \cdot u^{\overline{k}} \cdot u^{\overline{l}}) \ ,
\end{gathered} \tag{26}
$$

$$
e_{,k} = e_k e \ , \qquad\qquad \theta = \theta_o - \theta_1 \ . \tag{27}
$$

The invariant θ is the *expansion velocity,* the invariant θ_o is the *torsion-free expansion velocity,* the invariant θ_1 is the *expansion velocity induced by the torsion,* the antisymmetric tensor ω is the *rotation (vortex) velocity* tensor (rotation velocity, vortex velocity),

$$
\omega = h_u(k_a)h_u = h_u(s)h_u - h_u(q)h_u = S - Q \ , \tag{28}
$$

$$
\begin{gathered}
s = \tfrac{1}{2} \cdot (u^k{}_{;m} \cdot g^{ml} - u^l{}_{;m} \cdot g^{mk}) \cdot e_k \wedge e_l \ , \\
e_k \wedge e_l = \tfrac{1}{2} \cdot (e_k \otimes e_l - e_l \otimes e_k) \ ,
\end{gathered} \tag{29}
$$

$$
\begin{gathered}
q = \tfrac{1}{2} \cdot (T_{mn}{}^{k} \cdot g^{ml} - T_{mn}{}^{l} \cdot g^{mk}) \cdot u^n \cdot e_k \wedge e_l \ , \\
S = h_u(s)h_u \ , \qquad\qquad Q = h_u(q)h_u \ .
\end{gathered} \tag{30}
$$

The antisymmetric tensor S is the *torsion-free rotation (vortex) velocity* tensor, and the antisymmetric tensor Q is the *rotation (vortex) velocity* tensor *induced by the torsion.*

By means of the expressions for σ, ω and θ the deformation velocity tensor can be written in two parts

$$
\begin{gathered}
d = d_o - d_1 \ , \\
d_o = {}_sE + S + \tfrac{1}{n-1} \cdot \theta_o \cdot h_u \ , \\
d_1 = {}_sP + Q + \tfrac{1}{n-1} \cdot \theta_1 \cdot h_u \ ,
\end{gathered} \tag{31}
$$

where d_o is the *torsion-free deformation velocity* tensor and d_1 is the *deformation velocity* tensor *induced by the torsion.* For the case of V_n-spaces $d_1 = 0$ (${}_sP = 0$, $Q = 0$, $\theta_1 = 0$).

If we use the explicit form of the tensor k_s from (23) and (24) , we can find the relations

$$
\begin{aligned}
k_s &= \varepsilon - m = (\varepsilon^{kl} - m^{kl}) \cdot \partial_k \otimes \partial_l \ , \\
\varepsilon^{kl} - m^{kl} &= \frac{1}{2} \cdot [u^k{}_{;n} \cdot g^{nl} + u^l{}_{;n} \cdot g^{nk} - (T_{mn}{}^{k} \cdot g^{ml} + T_{mn}{}^{l} \cdot g^{mk}) \cdot u^n] = \\
&= \frac{1}{2} \cdot [u^k{}_{;n} \cdot g^{nl} + u^l{}_{;n} \cdot g^{nk} - T_{mn}{}^{k} \cdot g^{ml} \cdot u^n - T_{mn}{}^{l} \cdot g^{mk} \cdot u^n] \ .
\end{aligned} \tag{32}
$$

On the other side,

$$
\begin{aligned}
\nabla_u \overline{g} &= g^{ij}{}_{;k} \cdot u^k \cdot \partial_i . \partial_j \ , \qquad \pounds_u \overline{g} = (\pounds_u g^{ij}) \cdot \partial_i . \partial_j \ , \\
\pounds_u g^{ij} &= g^{ij}{}_{;k} \cdot u^k - \\
&\quad -[u^i{}_{;l} \cdot g^{lj} + u^j{}_{;l} \cdot g^{il} - T_{lk}{}^{i} \cdot u^k \cdot g^{lj} - T_{lk}{}^{j} \cdot u^k \cdot g^{il}] \ , \\
\pounds_u g^{kl} &= g^{kl}{}_{;n} \cdot u^n - \\
&\quad -[u^k{}_{;n} \cdot g^{nl} + u^l{}_{;n} \cdot g^{nk} - T_{mn}{}^{k} \cdot u^n \cdot g^{ml} - T_{mn}{}^{l} \cdot u^n \cdot g^{km}] \ , \\
g^{kl}{}_{;n} \cdot u^n - \pounds_u g^{kl} &= \\
&= u^k{}_{;n} \cdot g^{nl} + u^l{}_{;n} \cdot g^{nk} - T_{mn}{}^{k} \cdot u^n \cdot g^{ml} - T_{mn}{}^{l} \cdot u^n \cdot g^{km} \ .
\end{aligned} \tag{33}
$$

Therefore,

$$
\begin{aligned}
\varepsilon^{kl} - m^{kl} &= \frac{1}{2} \cdot (g^{kl}{}_{;n} \cdot u^n - \pounds_u g^{kl}) \ , \\
\varepsilon - m &= \frac{1}{2} \cdot (\nabla_u \overline{g} - \pounds_u \overline{g}) = k_s \ , \\
h_u(k_s)h_u &= \frac{1}{2} \cdot h_u(\nabla_u \overline{g} - \pounds_u \overline{g})h_u \ .
\end{aligned} \tag{34}
$$

By the use of the last relations the shear velocity tensor σ and the expansion velocity invariant θ can also be written in the form

$$\sigma = \frac{1}{2} \cdot \{h_u(\nabla_u \overline{g} - \pounds_u \overline{g})h_u - \frac{1}{n-1} \cdot (h_u[\nabla_u \overline{g} - \pounds_u \overline{g}]) \cdot h_u\} = \quad (35)$$

$$= \frac{1}{2} \cdot \{h_{i\overline{k}} \cdot (g^{kl}{}_{;m} \cdot u^m - \pounds_u g^{kl}) \cdot h_{\overline{l}j} - \quad (36)$$
$$- \frac{1}{n-1} \cdot h_{\overline{kl}} \cdot (g^{kl}{}_{;m} \cdot u^m - \pounds_u g^{kl}) \cdot h_{ij}\} \cdot e^i.e^j \ .$$

$$\theta = \tfrac{1}{2} \cdot h_u[\nabla_u \overline{g} - \pounds_u \overline{g}] = \tfrac{1}{2} \cdot [\nabla_{\overline{g}} u + T(u, \overline{g})] =$$
$$= \tfrac{1}{2} \cdot h_{\overline{ij}} \cdot (g^{ij}{}_{;k} \cdot u^k - \pounds_u g^{ij}) \ , \quad (37)$$

where

$$\frac{1}{2} \cdot \overline{g}[h_u(\nabla_u \overline{g} - \pounds_u \overline{g})h_u] = \frac{1}{2} \cdot h_u[\nabla_u \overline{g} - \pounds_u \overline{g}] \ . \quad (38)$$

The main result of the above considerations can be summarized in the following proposition:

Proposition 1 *The covariant vector field $g({}_{rel}v) = h_u(\nabla_u \xi)$ can be written in the forms:*

$$h_u(\nabla_u \xi) = h_u(\frac{l}{e} \cdot a - \pounds_\xi u) + d(\xi) =$$

$$= h_u(\frac{l}{e} \cdot a - \pounds_\xi u) + \sigma(\xi) + \omega(\xi) + \frac{1}{n-1} \cdot \theta \cdot h_u(\xi) \ .$$

The physical interpretation of the velocity tensors d, σ, ω and of the invariant θ for the case of V_4-spaces [4], , [5] , can also be extended for $(\overline{L}_4, g)$-spaces (see Fig. 1). In this case the torsion plays an equivalent role in the velocity tensors as the covariant derivative. It is easy to see that the existence of some kinematic characteristics $({}_sP,\ Q,\ \theta_1)$ depends on the existence of the torsion tensor field. They vanish if it is equal to zero (e.g. in V_n-spaces). On the other side, the kinematic characteristics induced by the torsion can compensate the result of the action of the torsion-free kinematic characteristics. For $d = 0$, $\sigma = 0$, $\omega = 0$, $\theta = 0$ we could have the relations $d_0 = d_1$, ${}_sE = {}_sP$, $S = Q$, $\theta_0 = \theta_1$ respectively leading to vanishing the relative velocity ${}_{rel}v$ under the additional conditions $g(u, \xi) = l = 0$ and $\pounds_\xi u = 0$.

The condition $\pounds_\xi u = 0 = [\xi, u]$ for the contravariant vector fields ξ and u induces a family of two dimensional sub manifolds of M. On these sub manifolds, one can choose the parameters of the integral curves of the two vector fields ξ and u as co-ordinates. This statement could be easily proved by the use of the relation

$$[\xi, u] = (\xi^i \cdot u^j{}_{,i} - u^i \cdot \xi^j{}_{,i}) \cdot .\partial_j = 0 \ . \quad (39)$$

For a two parametric congruence of curves (not intersecting curves) in M

$$x^i = x^i(\lambda, \tau) \quad (40)$$

with

$$\xi := \frac{\partial}{\partial \lambda} = \frac{\partial x^i}{\partial \lambda} \cdot \partial_i = \xi^i \cdot \partial_i \ , \qquad u^i := \frac{\partial}{\partial \tau} = \frac{\partial x^i}{\partial \tau} \cdot \partial_i = u^i \cdot \partial_i \ , \quad (41)$$

the integrability condition for the co-ordinates x^i along λ and τ follows in the form

$$u^j{}_{,i} \cdot \xi^i = \frac{\partial^2 x^j}{\partial \lambda \partial \tau} = \frac{\partial^2 x^j}{\partial \tau \partial \lambda} = \xi^j{}_{,i} \cdot u^i \ . \quad (42)$$

The last expression leads to a solution of the equations for $x^i(\tau, \lambda)$

$$dx^i = \frac{\partial x^i}{\partial \lambda} \cdot d\lambda + \frac{\partial x^i}{\partial \tau} \cdot d\tau = \xi^i \cdot d\lambda + u^i \cdot d\tau \ . \quad (43)$$

A one to one correspondence could be established between two of the co-ordinates x^i (for instance, for $i = 1, 2$) and the parameters λ and τ on the basis of the relations

$$x^{a'} \simeq (\lambda, \tau) \ , \quad x^a \simeq (x^1, x^2) \ , \quad x^{a'} = x^{a'}(x^a) \ , \quad x^a = x^a(x^{a'}) \ ,$$

$$dx^{a'} = \frac{\partial x^{a'}}{\partial x^a} \cdot dx^a \ , \qquad dx^a = \frac{\partial x^a}{\partial x^{a'}} \cdot dx^{a'} \ . \quad (44)$$

2.2 Physical interpretation of the notion of relative velocity

2.2.1 Acceleration

Let us now consider the change of the velocity u during the motion of a material point from the point P with co-ordinates $x^i(\tau_0, \lambda_0^a)$ to the point P_1 with the co-ordinates $x^i(\tau_0 + d\tau, \lambda_0^a)$. If we wish to express $u(\tau_0 + d\tau, \lambda_0^a)$ by means of $u(\tau_0, \lambda_0^a)$ we could use the exponent $\exp[d\tau \cdot \nabla_u)]$ of the covariant differential operator $\nabla_u = D/d\tau$ with $u = d/d\tau$

$$u_{(\tau_0+d\tau,\lambda_0^a)} = u_{(\tau_0,\lambda_0^a)} + d\tau \cdot \left(\frac{Du}{d\tau}\right)_{(\tau_0,\lambda_0^a)} + \frac{1}{2!} \cdot d\tau^2 \cdot \left(\frac{D^2u}{d\tau^2}\right)_{(\tau_0,\lambda_0^a)} + \cdots \ . \tag{45}$$

Up to the first order of $d\tau$ we have

$$u_{(\tau_0+d\tau,\lambda_0^a)} = u_{(\tau_0,\lambda_0^a)} + d\tau \cdot \left(\frac{Du}{d\tau}\right)_{(\tau_0,\lambda_0^a)} \ . \tag{46}$$

Therefore, the change of the velocity u from the point P with $x^i(\tau_0, \lambda_0^a)$ to the point P_1 with $x^i(\tau_0 + d\tau, \lambda_0^a)$ is

$$u_{(\tau_0+d\tau,\lambda_0^a)} - u_{(\tau_0,\lambda_0^a)} = d\tau \cdot \left(\frac{Du}{d\tau}\right)_{(\tau_0,\lambda_0^a)} \ . \tag{47}$$

The covariant derivative

$$\left(\frac{Du}{d\tau}\right)_{(\tau_0,\lambda_0^a)} = (\nabla_u u)_{(\tau_0,\lambda_0^a)} := a_{(\tau_0,\lambda_0^a)} = \lim_{d\tau \to 0} \frac{u_{(\tau_0+d\tau,\lambda_0^a)} - u_{(\tau_0,\lambda_0^a)}}{d\tau} \ , \tag{48}$$

with $u = d/d\tau = u^i \cdot \partial_i$ and $a = \nabla_u u$, can be interpreted as the acceleration a of a material point at the point P with $x^i(\tau_0, \lambda_0^a)$ of the curve $x^i(\tau, \lambda_0^a = \text{const.})$.

In analogous way, the acceleration of a material point during its motion (transport) along a curve $x^i(\tau_0 = \text{const.}, \lambda^a)$ could be found in the form

$$\left(\frac{D\xi_{(a)\perp}}{d\lambda^a}\right)_{(\tau_0,\lambda_0^a)} = \left(\nabla_{\xi_{(a)\perp}} \xi_{(a)\perp}\right)_{(\tau_0,\lambda_0^a)} = \lim_{d\lambda \to 0} \frac{\xi_{(a)\perp(\tau_0,\lambda_0^a+d\lambda^a)} - \xi_{(a)\perp(\tau_0,\lambda_0^a)}}{d\lambda^a} \ , \tag{49}$$

where $\xi_{(a)\perp} = d/d\lambda^a$. The last expression shows the difference between the velocities along a curve $x^i(\tau_0 = \text{const.}, \lambda^a)$ at the two different points P_2 with $x^i(\tau_0, \lambda_0^a + d\lambda^a)$ and P with $x^i(\tau_0, \lambda_0^a)$. Usually, the velocity u is interpreted as the velocity of the material points (elements) in the flow. The set of vectors $\overline{\xi}_{(a)}$ $(a = 1, ..., n-1)$ determines a cross-section of a flow in a neighborhood of a given point. Since $\nabla_{\xi_{(a)\perp}} \xi_{(a)\perp}$ is the first curvature vector of the curve $x^i(\tau_0, \lambda^a)$, it could be interpreted as a measure for the deviation of an infinitesimal cross-section of the flow with $\nabla_{\xi_{(a)\perp}} \xi_{(a)\perp} \neq 0$, orthogonal to u, from an auto-parallel (constructed by auto-parallel lines of $\xi_{(a)\perp}$) infinitesimal cross-section with $\nabla_{\xi_{(a)\perp}} \xi_{(a)\perp} = 0$, orthogonal to u.

On the other side, we can consider

(a) the change of the vector $\xi_{(a)\perp}$ along the curve $x^i(\tau, \lambda_0^a = \text{const.})$ and

(b) the change of the vector u along the curve $x^i(\tau_0 = \text{const.}, \lambda^a)$.

2.2.2 Relative velocity

(a) In the first case,

$$\left(\frac{D\xi_{(a)\perp}}{d\tau}\right)_{(\tau_0,\lambda_0^a)} = \left(\nabla_u \xi_{(a)\perp}\right)_{(\tau_0,\lambda_0^a)} = \lim_{d\tau \to 0} \frac{\xi_{(a)\perp(\tau_0+d\tau,\lambda_0^a)} - \xi_{(a)\perp(\tau_0,\lambda_0^a)}}{d\tau} \ . \tag{50}$$

The vector $\nabla_u \xi_{(a)\perp}$ has two components with respect to the vector u: one collinear to u and one orthogonal to u, i.e.

$$\begin{aligned} \frac{D\xi_{(a)\perp}}{d\tau} &= \nabla_u \xi_{(a)\perp} = \frac{\overline{l}_a}{e} \cdot u + {}_{rel}v_{(a)} \ , \\ \overline{l}_a &: = g(u, \nabla_u \xi_{(a)\perp}) \quad , \quad e = g(u,u) \neq 0 \quad , \\ g(u, {}_{rel}v_{(a)}) &= 0 \ , \quad {}_{rel}v_{(a)} = \overline{g}[h_u(\nabla_u \xi_{(a)\perp})] = \overline{g}[h_u(\frac{D\xi_{(a)\perp}}{d\tau})] \ . \end{aligned} \tag{51}$$

The set of the infinitesimal vectors $\{\overline{\xi}_{(a)\perp}\}$ determines a tangential subspace at the point $x^i(\tau_0, \lambda_0)$, orthogonal to the vector u. This subspace intersect the flow in such a way that a flat cross-section appears, orthogonal to u . All material points (elements) of the flow lying at this cross-section have one and the same proper time τ_0 (if τ is interpreted as proper time). The vector $\xi_{(a)\perp}$ was interpreted as the velocity along the line $x^i(\tau_0, \lambda^a)$. If we consider instead of $\xi_{(a)\perp}$ the infinitesimal vector $\overline{\xi}_{(a)\perp} := d\lambda^a \cdot \xi_{(a)\perp}$ (there is no summation over a) then the flat cross-section coincides with the cross-section determined by the points $\{x^i(\tau_0, \lambda_0^a + d\lambda^a), \quad a = 1, ..., n-1\}$. The infinitesimal vectors $\overline{\xi}_{(a)\perp}$, $a = 1, ..., n-1$, are equal to the difference between the co-ordinates of the point P with $x^i(\tau_0, \lambda_0^a)$ and the points with co-ordinates $x^i(\tau_0, \lambda_0^a + d\lambda^a$, $a = 1, ..., n-1)$. The change of the vectors $\overline{\xi}_{(a)\perp}$, determined by the parts $\overline{g}[h_u(\nabla_u \xi_{(a)\perp})]$ orthogonal to u and lying at the flat cross-section [when it moves along the curve $x^i(\tau, \lambda_0^a)$], is described by

$$\left({}_{rel}\overline{v}_{(a)}\right)_{(\tau_0,\lambda_0^a)} = \left(\overline{g}[h_u(\nabla_u \overline{\xi}_{(a)\perp})]\right)_{(\tau_0,\lambda_0^a)} = \left(\overline{g}[h_u(\frac{D\overline{\xi}_{(a)\perp}}{d\tau})]\right)_{(\tau_0,\lambda_0^a)} \quad . \tag{52}$$

Therefore, we can interpret the vector ${}_{rel}\overline{v}_{(a)(\tau_0,\lambda_0^a)}$ as the *relative velocity vector* or *relative velocity* of material points with co-ordinates $x^i(\tau_0, \lambda_0^a + d\lambda^a)$ with respect to the point with co-ordinates $x^i(\tau_0, \lambda_0^a)$. Let us now determine the relation between ${}_{rel}\overline{v}$ and the total velocity ${}_{total}\overline{v}$ defined at the point P with $x^i(\tau_0, \lambda_0^a)$ as

$${}_{total}\overline{v}_{(a)(\tau_0,\lambda_0^a)} := \lim_{d\tau \to 0} \frac{x^i(\tau_0 + d\tau, \lambda_0^a + d\lambda^a) - x^i(\tau_0, \lambda_0^a)}{d\tau} \quad . \tag{53}$$

Since up to the second order of $d\tau$ and $d\lambda^a$ we have

$$\begin{aligned} x^i(\tau_0 + d\tau, \lambda_0^a + d\lambda^a) &= x^i(\tau_0 + d\tau, \lambda_0^a) + d\lambda^a \cdot \left(\frac{\partial x^i}{\partial \lambda^a}\right)_{(\tau_0+d\tau,\lambda_0^a)} = \\ &= x^i(\tau_0 + d\tau, \lambda_0^a) + \overline{\xi}^{\ i}_{(a)\perp(\tau_0+d\tau,\lambda_0^a)} = \\ &= x^i(\tau_0, \lambda_0^a) + d\tau \cdot \left(\frac{\partial x^i}{\partial \tau}\right)_{(\tau_0,\lambda_0^a)} + \overline{\xi}^{\ i}_{(a)\perp(\tau_0,\lambda_0^a)} + \\ &\quad + d\tau \cdot \left(\frac{D\overline{\xi}^i_{(a)\perp}}{d\tau}\right)_{(\tau_0,\lambda_0^a)} = \end{aligned}$$

$$= x^i(\tau_0, \lambda_0^a) + \overline{u}^i{}_{(\tau_0,\lambda_0^a)} + \overline{\xi}^{\ i}_{(a)\perp(\tau_0,\lambda_0^a)} + d\tau \cdot \left(\frac{D\overline{\xi}^i_{(a)\perp}}{d\tau}\right)_{(\tau_0,\lambda_0^a)} \quad , \tag{54}$$

we can express the last term in the last relation as

$$\frac{D\overline{\xi}^i_{(a)\perp}}{d\tau} = \frac{\widehat{l}_a}{e} \cdot u^i + {}_{rel}\overline{v}^i_{(a)} \ , \qquad \widehat{l}_a := g(u, \nabla_u \overline{\xi}_{(a)\perp}) \quad , \quad g(u, {}_{rel}\overline{v}_{(a)}) = 0 \ . \tag{55}$$

Therefore,

$$\begin{aligned} {}_{total}\overline{v}_{(\tau_0,\lambda_0^a)} &= \lim_{d\tau \to 0} \frac{x^i(\tau_0 + d\tau, \lambda_0^a + d\lambda^a) - x^i(\tau_0, \lambda_0^a)}{d\tau} = \\ &= \left[(1 + \frac{\widehat{l}_a}{e}) \cdot u^i\right]_{(\tau_0,\lambda_0^a)} + {}_{rel}\overline{v}^i_{(a)(\tau_0,\lambda_0^a)} \ , \qquad \frac{d\lambda}{d\tau} = 0 \quad , \end{aligned} \tag{56}$$

or for every point at a curve $x^i(\tau, \lambda^a) \subset M$ the local total velocity is

$${}_{total}\overline{v}_{(a)} = (1 + \frac{\widehat{l}_a}{e}) \cdot u + {}_{rel}\overline{v}_{(a)} \quad , \tag{57}$$

and the local relative velocity ${}_{rel}\overline{v}_{(a)}$ of the material point (element) in a flow is

$${}_{rel}\overline{v}_{(a)} = {}_{total}\overline{v}_{(a)} - (1 + \frac{\widehat{l}_a}{e}) \cdot u \quad . \tag{58}$$

To find a more exact physical explanation of the structure of the relative velocity vector ${}_{rel}v$ we should consider now the length of a vector field, expressed by the use of kinematic characteristics of a flow. In the further consideration we use the vector $u \in T(M)$ instead of the vector $\overline{u} = d\tau \cdot u \in T(M)$ as an infinitesimal vector at a curve $x^i(\tau, \lambda_0^a)$ identified with a line segment of the curve.

2.3 Length of a vector field, expressed by the use of the kinematic characteristics of a flow

Let us now consider the change of the length of a vector $\xi_{(a)\perp}$ $(a = 1, ..., n-1)$ transported from point P with co-ordinates $x^i(\tau_0, \lambda_0^a)$ to the point P_1 with co-ordinates $x^i(\tau_0 + d\tau, \lambda_0^a)$. The length $l_{\xi_{(a)\perp}}$ of the vector $\xi_{(a)\perp}$ is defined as

$$g(\xi_{(a)\perp}, \xi_{(a)\perp}) = g_{\overline{ij}} \cdot \xi^i_{(a)\perp} \cdot \xi^j_{(a)\perp} = \pm\, l^2_{\xi_{(a)\perp}} \quad . \tag{59}$$

At the point P with $x^i(\tau_0, \lambda_0^a)$ the vector $\xi_{(a)\perp}$ will have the length

$$\left[g(\xi_{(a)\perp}, \xi_{(a)\perp})\right]_{(\tau_0,\lambda_0^a)} = \left[g_{\overline{ij}} \cdot \xi^i_{(a)\perp} \cdot \xi^j_{(a)\perp}\right]_{(\tau_0,\lambda_0^a)} = \pm \left[l^2_{\xi_{(a)\perp}}\right]_{(\tau_0,\lambda_0^a)} \quad . \tag{60}$$

At the point P_1 with $x^i(\tau_0 + d\tau, \lambda_0^a)$ the vector $\xi_\perp$ will have the length

$$\left[g(\xi_{(a)\perp}, \xi_{(a)\perp})\right]_{(\tau_0+d\tau,\lambda_0^a)} = \left[g(\xi_{(a)\perp}, \xi_{(a)\perp})\right]_{(\tau_0,\lambda_0^a)} + d\tau \cdot \left[\frac{d[g(\xi_{(a)\perp}, \xi_{(a)\perp})]}{d\tau}\right]_{(\tau_0,\lambda_0^a)} \quad . \tag{61}$$

Since

$$\frac{d[g(\xi_{(a)\perp}, \xi_{(a)\perp})]}{d\tau} = \frac{D[g(\xi_{(a)\perp}, \xi_{(a)\perp})]}{d\tau} = \nabla_u [g(\xi_{(a)\perp}, \xi_{(a)\perp})] \quad , \tag{62}$$

we can represent the last expression by the use of the kinematic characteristics of the relative velocity.

$$\begin{aligned} \frac{D[g(\xi_{(a)\perp}, \xi_{(a)\perp})]}{d\tau} &= \nabla_u [g(\xi_{(a)\perp}, \xi_{(a)\perp})] = (\nabla_u g)(\xi_{(a)\perp}, \xi_{(a)\perp}) + 2 \cdot g(\xi_{(a)\perp}, \nabla_u \xi_{(a)\perp}) = \\ &= u[g(\xi_{(a)\perp}, \xi_{(a)\perp})] = \frac{d}{d\tau}[g(\xi_{(a)\perp}, \xi_{(a)\perp})] \quad . \end{aligned} \tag{63}$$

The vector $\nabla_u \xi_{(a)\perp}$ could be represented in the form

$$\nabla_u \xi_{(a)\perp} = \frac{g(\nabla_u \xi_{(a)\perp}, u)}{g(u,u)} \cdot u + {}_{rel}v_{(a)} = \frac{\overline{l}_a}{e} \cdot u + {}_{rel}v_{(a)} \quad , \tag{64}$$

where

$${}_{rel}v_{(a)} = \overline{g}[d(\xi_{(a)\perp}] \quad . \tag{65}$$

The tensor d with $d(u) = (u)(d) = 0$ is the deformation velocity tensor. The relative velocity between the material points of the flow is related to the deformation of a cross-section of the flow (determined by $\{\xi_{(a)} : a = 1, \cdots, n-1\}$ along a line of the flow with tangent vector u.

The covariant derivative of $g(\xi_{(a)\perp}, \xi_{(a)\perp})$ (identical to the ordinary derivative) along the vector u could be written now as

$$\begin{aligned} \frac{D[g(\xi_{(a)\perp}, \xi_{(a)\perp})]}{d\tau} &= (\nabla_u g)(\xi_{(a)\perp}, \xi_{(a)\perp}) + 2 \cdot g(\xi_{(a)\perp}, \frac{\overline{l}_a}{e} \cdot u + {}_{rel}v_{(a)}) = \\ &= (\nabla_u g)(\xi_{(a)\perp}, \xi_{(a)\perp}) + 2 \cdot \frac{\overline{l}_a}{e} \cdot g(\xi_{(a)\perp}, u) + 2 \cdot g(\xi_{(a)\perp}, {}_{rel}v_{(a)}) \quad . \end{aligned} \tag{66}$$

Since $g(\xi_{(a)\perp}, u) = 0$, the second term at the right side vanishes and we have

$$\frac{D[g(\xi_{(a)\perp}, \xi_{(a)\perp})]}{d\tau} = (\nabla_u g)(\xi_{(a)\perp}, \xi_{(a)\perp}) + 2 \cdot g(\xi_{(a)\perp}, {}_{rel}v_{(a)}) \quad . \tag{67}$$

Let us now represent $\nabla_u g$ by means of the projective metric h_u corresponding to the vector field u.

2.3.1 Representation of $\nabla_u g$ by means of the projective metric h_u

Since $h_u = g - (1/e) \cdot g(u) \otimes g(u)$, the following relations can be found:

$$\begin{aligned} \nabla_u g &= g(\overline{g})(\nabla_u g)(\overline{g})g = \left(h_u + \frac{1}{e} \cdot g(u) \otimes g(u)\right)(\overline{g})(\nabla_u g)(\overline{g})g = \\ &= h_u(\overline{g})(\nabla_u g)(\overline{g})g + \frac{1}{e} \cdot g(u) \otimes u(\nabla_u g)(\overline{g})g = \end{aligned}$$

$$
\begin{aligned}
&= h_u(\overline{g})(\nabla_u g)(\overline{g})\left(h_u + \frac{1}{e}\cdot g(u)\otimes g(u)\right) + \\
&\quad + \frac{1}{e}\cdot g(u)\otimes u(\nabla_u g)(\overline{g})\left(h_u + \frac{1}{e}\cdot g(u)\otimes g(u)\right) \\
&= h_u(\overline{g})(\nabla_u g)(\overline{g})h_u + \frac{1}{e}\cdot h_u(\overline{g})(\nabla_u g)(u)\otimes g(u) + \\
&\quad + \frac{1}{e}\cdot g(u)\otimes u(\nabla_u g)(\overline{g})(h_u) + \frac{1}{e^2}\cdot(\nabla_u g)(u,u)\cdot g(u)\otimes g(u)\ ,
\end{aligned}
\tag{68}
$$

where

$$
\begin{aligned}
g(u)\overline{g} &= u \quad , \quad h_u(\overline{g})(\nabla_u g)(u) = (u)(\nabla_u g)(\overline{g})h_u \quad , \\
(u)(\nabla_u g)(u) &= (\nabla_u g)(u,u) \quad .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\nabla_u g &= h_u(\overline{g})(\nabla_u g)(\overline{g})h_u + \\
&\quad + \frac{1}{e}\cdot[h_u(\overline{g})(\nabla_u g)(u)\otimes g(u) + g(u)\otimes h_u(\overline{g})(\nabla_u g)(u)] + \\
&\quad + \frac{1}{e^2}\cdot(\nabla_u g)(u,u)\cdot g(u)\otimes g(u)\ .
\end{aligned}
\tag{69}
$$

2.3.2 Explicit form of the change of the length of the vector fields $\xi_{(a)\perp}$

If we use the relations

$$
\begin{aligned}
\overline{g}(\nabla_u g)\overline{g} &= -\nabla_u\overline{g} \quad , \qquad (g(u)\otimes g(u))\,(\xi_{(a)\perp},\xi_{(a)\perp}) = 0 \quad , \\
&\quad\ (h_u(\overline{g})(\nabla_u g)(u)\otimes g(u))\,(\xi_{(a)\perp},\xi_{(a)\perp}) \\
&= (h_u(\overline{g})(\nabla_u g)(u))\,(\xi_{(a)\perp})\cdot g(u,\xi_{(a)\perp}) = 0 \quad , \\
&\quad\ (g(u)\otimes h_u(\overline{g})(\nabla_u g)(u))\,(\xi_{(a)\perp},\xi_{(a)\perp}) \\
&= g(u,\xi_{(a)\perp})\cdot(h_u(\overline{g})(\nabla_u g)(u))\,(\xi_{(a)\perp}) = 0\ ,
\end{aligned}
\tag{70}
$$

we can find some useful expressions leading to the explicit form of the change of the length of the vector fields $\xi_{(a)\perp}$.

For $(\nabla_u g)(\xi_{(a)\perp},\xi_{(a)\perp})$, we obtain

$$
\begin{aligned}
(\nabla_u g)(\xi_{(a)\perp},\xi_{(a)\perp}) &= (h_u(\overline{g})(\nabla_u g)(\overline{g})h_u)\,(\xi_{(a)\perp},\xi_{(a)\perp}) = \\
&= -\ (h_u(\nabla_u\overline{g})h_u)\,(\xi_{(a)\perp},\xi_{(a)\perp}) \quad .
\end{aligned}
\tag{71}
$$

On the other side,

$$
\begin{aligned}
\overline{g}\,[h_u(\nabla_u\overline{g})h_u] &= g^{\overline{ij}}\cdot h_{ik}\cdot g^{\overline{kl}}{}_{;n}\cdot u^n\cdot h_{lj} = \\
&= g^{\overline{ij}}\cdot h_{i\overline{k}}\cdot g^{kl}{}_{;n}\cdot u^n\cdot h_{\overline{l}j} = \\
&= h_{ki}\cdot g^{\overline{ij}}\cdot h_{jl}\cdot g^{\overline{kl}}{}_{;n}\cdot u^n = \\
&= (h_u(\overline{g})h_u)\,[\nabla_u\overline{g}] \quad ,
\end{aligned}
\tag{72}
$$

$$
\begin{aligned}
[g(u)]\overline{g} &= \overline{g}[g(u)] = u \quad , \qquad h_u(\overline{g})h_u = g(\overline{g})h_u = h_u \quad , \\
\overline{g}[h_u(\nabla_u\overline{g})h_u] &= h_u[\nabla_u\overline{g}] = h_{\overline{ij}}\cdot g^{ij}{}_{;n}\cdot u^n \quad .
\end{aligned}
\tag{73}
$$

It follows now for $(\nabla_u g)(\xi_{(a)\perp},\xi_{(a)\perp})$

$$
\begin{aligned}
(\nabla_u g)(\xi_{(a)\perp},\xi_{(a)\perp}) &= -[h_u(\nabla_u\overline{g})h_u - \frac{1}{n-1}\cdot(h_u[\nabla_u\overline{g}])\cdot h_u](\xi_{(a)\perp},\xi_{(a)\perp}) - \\
&\quad -\frac{1}{n-1}\cdot(h_u[\nabla_u\overline{g}])\cdot h_u(\xi_{(a)\perp},\xi_{(a)\perp}) \quad .
\end{aligned}
\tag{74}
$$

After introducing the abbreviations:

$$ {}_{\nabla}\sigma \;:\; = \frac{1}{2} \cdot [h_u(\nabla_u \overline{g})h_u - \frac{1}{n-1} \cdot (h_u[\nabla_u \overline{g}]) \cdot h_u] \quad , \tag{75} $$

$$ {}_{\nabla}\theta \;:\; = \frac{1}{2} \cdot h_u[\nabla_u \overline{g}] \quad , \tag{76} $$

we obtain for $(\nabla_u g)(\xi_{(a)\perp}, \xi_{(a)\perp})$

$$ (\nabla_u g)(\xi_{(a)\perp}, \xi_{(a)\perp}) = -2 \cdot {}_{\nabla}\sigma(\xi_{(a)\perp}, \xi_{(a)\perp}) - \frac{2}{n-1} \cdot_{\nabla} \theta \cdot h_u(\xi_{(a)\perp}, \xi_{(a)\perp}) \quad . \tag{77} $$

We can now find the explicit form of $\nabla_u[g(\xi_{(a)\perp}, \xi_{(a)\perp})]$ by the use of the last relations

$$ \begin{aligned} \frac{D}{d\tau}[g(\xi_{(a)\perp}, \xi_{(a)\perp})] &= (\nabla_u g)(\xi_{(a)\perp}, \xi_{(a)\perp}) + 2 \cdot g(\xi_{(a)\perp}, {}_{rel}v_{(a)}) = \\ &= -2 \cdot {}_{\nabla}\sigma(\xi_{(a)\perp}, \xi_{(a)\perp}) - \frac{2}{n-1} \cdot_{\nabla} \theta \cdot h_u(\xi_{(a)\perp}, \xi_{(a)\perp}) + \\ &\quad +2 \cdot g(\xi_{(a)\perp}, {}_{rel}v_{(a)}) \ . \end{aligned} \tag{78} $$

On the other side,

$$ \begin{aligned} g(\xi_{(a)\perp}, {}_{rel}v_{(a)}) &= g(\xi_{(a)\perp}, \overline{g}[d(\xi_{(a)\perp}) = \\ &= g_{\overline{ij}} \cdot \xi^i_{(a)\perp} \cdot g^{jk} \cdot d_{\overline{kl}} \cdot \xi^l_{(a)\perp} = \\ &= g_{\overline{ij}} \cdot g^{jk} \cdot \xi^i_{(a)\perp} \cdot d_{\overline{kl}} \cdot \xi^l_{(a)\perp} = \\ &= d_{\overline{kl}} \cdot \xi^k_{(a)\perp} \cdot \xi^l_{(a)\perp} = d(\xi^k_{(a)\perp}, \xi^l_{(a)\perp}) \quad . \end{aligned} \tag{79} $$

Therefore,

$$ \begin{aligned} \frac{D}{d\tau}[g(\xi_{(a)\perp}, \xi_{(a)\perp})] &= -2 \cdot {}_{\nabla}\sigma(\xi_{(a)\perp}, \xi_{(a)\perp}) - \frac{2}{n-1} \cdot_{\nabla} \theta \cdot h_u(\xi_{(a)\perp}, \xi_{(a)\perp}) + \\ &\quad +2 \cdot d(\xi^k_{(a)\perp}, \xi^l_{(a)\perp}) \ . \end{aligned} \tag{80} $$

The deformation velocity tensor d could be given in its explicit form as [6]:

$$ d = \sigma + \omega + \frac{1}{n-1} \cdot \theta \cdot h_u \ , \tag{81} $$

where

$$ \begin{aligned} \sigma &= \frac{1}{2} \cdot \{h_u(\nabla_u \overline{g})h_u - h_u(\pounds_u \overline{g})h_u - \\ &\quad - \frac{1}{n-1} \cdot (h_u[\nabla_u \overline{g}]) \cdot h_u + \frac{1}{n-1} \cdot (h_u[\pounds_u \overline{g}]) \cdot h_u\} \quad , \end{aligned} \tag{82} $$

$$ \theta = \frac{1}{2} \cdot h_u[\nabla_u \overline{g}] - \frac{1}{2} \cdot h_u[\pounds_u \overline{g}] \quad . \tag{83} $$

If we introduce the abbreviations

$$ {}_{\pounds}\sigma = \frac{1}{2} \cdot \{h_u(\pounds_u \overline{g})h_u - \frac{1}{n-1} \cdot (h_u[\pounds_u \overline{g}]) \cdot h_u\} \quad , \tag{84} $$

$$ {}_{\pounds}\theta = \frac{1}{2} \cdot h_u[\pounds_u \overline{g}] \quad , \tag{85} $$

it follows for the shear velocity tensor σ and for the expansion velocity invariant θ the expressions

$$ \sigma = {}_{\nabla}\sigma - {}_{\pounds}\sigma \quad , \quad \theta = {}_{\nabla}\theta - {}_{\pounds}\theta \quad . \tag{86} $$

Then we obtain the deformation velocity tensor d in the form

$$ d = {}_{\nabla}\sigma - {}_{\pounds}\sigma + \omega + \frac{1}{n-1} \cdot ({}_{\nabla}\theta - {}_{\pounds}\theta) \cdot h_u \quad , \tag{87} $$

and $\nabla_u[g(\xi_{(a)\perp},\xi_{(a)\perp})]$ could be written in the form

$$
\begin{aligned}
\frac{D}{d\tau}[g(\xi_{(a)\perp},\xi_{(a)\perp})] &= \frac{d}{d\tau}[g(\xi_{(a)\perp},\xi_{(a)\perp})] = \\
&= -2\cdot {}_{\nabla}\sigma(\xi_{(a)\perp},\xi_{(a)\perp}) - \frac{2}{n-1}\cdot {}_{\nabla}\theta\cdot h_u(\xi_{(a)\perp},\xi_{(a)\perp}) \;+ \\
&\quad +2\cdot {}_{\nabla}\,\sigma(\xi_{(a)\perp},\xi_{(a)\perp}) - 2\cdot {}_{\pounds}\sigma(\xi_{(a)\perp},\xi_{(a)\perp}) + \\
&\quad +2\cdot\omega(\xi_{(a)\perp},\xi_{(a)\perp}) + \frac{2}{n-1}\cdot {}_{\nabla}\theta\cdot h_u(\xi_{(a)\perp},\xi_{(a)\perp}) - \\
&\quad -\frac{2}{n-1}\cdot {}_{\pounds}\theta\cdot h_u(\xi_{(a)\perp},\xi_{(a)\perp}) \\
\frac{D}{d\tau}[g(\xi_{(a)\perp},\xi_{(a)\perp})] &= -2\cdot[{}_{\pounds}\sigma(\xi_{(a)\perp},\xi_{(a)\perp}) + \frac{1}{n-1}\cdot {}_{\pounds}\theta\cdot h_u(\xi_{(a)\perp},\xi_{(a)\perp})]\;\;, \qquad (88)
\end{aligned}
$$

where $\omega(\xi_{(a)\perp},\xi_{(a)\perp})=0$. Therefore, the change of the square $\pm\, l^2_{\xi_{(a)\perp}} = g(\xi_{(a)\perp},\xi_{(a)\perp}) = g_{\overline{ij}}\cdot\xi^i_{(a)\perp}\cdot\xi^j_{(a)\perp}$ of the length $l_{\xi_{(a)\perp}}$ along the curve $x^i(\tau,\lambda^a_0)$ is depending only on the shear and expansion velocity induced by the dragging along u and not on the transport along u.

Now we can determine the difference between the length of the vector $\xi_{(a)\perp}$ at the point P_1 with $x^i(\tau_0+d\tau,\lambda^a_0)$ and the length of the vector $\xi_{(a)\perp}$ at the point P with $x^i(\tau_0,\lambda^a_0)$:

$$
\begin{aligned}
&\left[g(\xi_{(a)\perp},\xi_{(a)\perp})\right]_{(\tau_0+d\tau,\lambda^a_0)} = \left[g(\xi_{(a)\perp},\xi_{(a)\perp})\right]_{(\tau_0,\lambda^a_0)} - \\
&-2\cdot d\tau\cdot\left[{}_{\pounds}\sigma(\xi_{(a)\perp},\xi_{(a)\perp}) + \frac{1}{n-1}\cdot {}_{\pounds}\theta\cdot h_u(\xi_{(a)\perp},\xi_{(a)\perp})\right]_{(\tau_0,\lambda^a_0)} \quad . \qquad (89)
\end{aligned}
$$

Since $h_u(\xi_{(a)\perp},\xi_{(a)\perp}) = g(\xi_{(a)\perp},\xi_{(a)\perp})$, we obtain

$$
\begin{aligned}
\left[g(\xi_{(a)\perp},\xi_{(a)\perp})\right]_{(\tau_0+d\tau,\lambda^a_0)} &= (1-\frac{2\cdot d\tau}{n-1}\cdot {}_{\pounds}\theta)\cdot\left[g(\xi_{(a)\perp},\xi_{(a)\perp})\right]_{(\tau_0,\lambda^a_0)} - \\
&\quad -2\cdot d\tau\cdot\left[\,{}_{\pounds}\sigma(\xi_{(a)\perp},\xi_{(a)\perp})\right]_{(\tau_0,\lambda^a_0)} \quad , \qquad (90)
\end{aligned}
$$

$$
\begin{aligned}
&\left\{\frac{D}{d\tau}[g(\xi_{(a)\perp},\xi_{(a)\perp})]\right\}_{(\tau_0,\lambda^a_0)} = \\
&= \lim_{d\tau\to 0}\frac{1}{d\tau}\cdot\left\{\left[g(\xi_{(a)\perp},\xi_{(a)\perp})\right]_{(\tau_0+d\tau,\lambda^a_0)} - \left[g(\xi_{(a)\perp},\xi_{(a)\perp})\right]_{(\tau_0,\lambda^a_0)}\right\} = \\
&= -2\cdot\left[{}_{\pounds}\sigma(\xi_{(a)\perp},\xi_{(a)\perp}) + \frac{1}{n-1}\cdot {}_{\pounds}\theta\cdot g(\xi_{(a)\perp},\xi_{(a)\perp})\right]_{(\tau_0,\lambda^a_0)} \quad . \qquad (91)
\end{aligned}
$$

If we introduce the set of unit vectors $\{n_{(a)}\}$ $(a=1,...,n-1)$

$$
n_{(a)} = \frac{\xi_{(a)\perp}}{l_{\xi_{(a)\perp}}} \quad , \qquad g(n_{(a)},n_{(a)}) = \frac{1}{l^2_{\xi_{(a)\perp}}}\cdot g(\xi_{(a)\perp},\xi_{(a)\perp}) = \pm 1 \;\; , \qquad (92)
$$

then

$$
\frac{Dl^2_{\xi_{(a)\perp}}}{d\tau} = \frac{dl^2_{\xi_{(a)\perp}}}{d\tau} = -2\cdot\left[{}_{\pounds}\sigma(l_{\xi_{(a)\perp}}\cdot n_{(a)}, l_{\xi_{(a)\perp}}\cdot n_{(a)}) + \frac{1}{n-1}\cdot {}_{\pounds}\theta\cdot l^2_{\xi_{(a)\perp}}\cdot g(n_{(a)},n_{(a)})\right] \;\; , \qquad (93)
$$

$$
\begin{aligned}
\frac{1}{l^2_{\xi_{(a)\perp}}}\cdot\frac{dl^2_{\xi_{(a)\perp}}}{d\tau} &= 2\cdot\frac{1}{l_{\xi_{(a)\perp}}}\cdot\frac{dl_{\xi_{(a)\perp}}}{d\tau} = \\
&= -2\cdot\left[{}_{\pounds}\sigma(n_{(a)},n_{(a)}) + \frac{1}{n-1}\cdot {}_{\pounds}\theta\cdot g(n_{(a)},n_{(a)})\right] = \\
&= -2\cdot\left[{}_{\pounds}\sigma(n_{(a)},n_{(a)}) \pm \frac{1}{n-1}\cdot {}_{\pounds}\theta\right] \quad , \qquad (94)
\end{aligned}
$$

$$
\frac{1}{l_{\xi_{(a)\perp}}}\cdot\frac{dl_{\xi_{(a)\perp}}}{d\tau} = -\left[{}_{\pounds}\sigma(n_{(a)},n_{(a)}) \pm \frac{1}{n-1}\cdot {}_{\pounds}\theta\right] \quad . \qquad (95)
$$

The invariant ${}_{\pounds}\theta$ is called *expansion velocity induced by the Lie differential operator* $\pounds_u$ *acting on the* contravariant metric $\overline{g}$. The invariant ${}_{\nabla}\theta$ is called *expansion velocity induced by the covariant differential operator* ∇_u acting on $\overline{g}$. It follows from the last two expressions that the length of the vector $\xi_{(a)\perp}$ does not depend on ${}_{\nabla}\theta$. The invariant $\theta = {}_{\nabla}\theta - {}_{\pounds}\theta$ is called expansion velocity. This notion is related to the fact that the change of the invariant volume element $d\omega$ along a given vector field u after a transport or a dragging along u could be expressed by means of ${}_{\nabla}\theta$ and ${}_{\pounds}\theta$ respectively. For $\nabla_u(d\omega)$ and $\pounds_u(d\omega)$ we have

$$\nabla_u(d\omega) = \frac{1}{2} \cdot \overline{g}[\nabla_u g] \cdot d\omega \quad , \quad \pounds_u(d\omega) = \frac{1}{2} \cdot \overline{g}[\pounds_u g] \cdot d\omega \quad . \tag{96}$$

On the other side, if we use the relations

$$\begin{aligned} \nabla_u \overline{g} &= -\overline{g}(\nabla_u g)\overline{g} \quad , \\ g[\overline{g}(\nabla_u g)\overline{g}] &= \overline{g}[\nabla_u g] \quad , \\ (g(u) \otimes g(u))\,[\overline{g}(\nabla_u g)\overline{g}] &= (\nabla_u g)(u,u) \quad , \end{aligned} \tag{97}$$

we can find the expression for ${}_{\nabla}\theta$ in the form

$$\begin{aligned} {}_{\nabla}\theta &= \frac{1}{2} \cdot h_u[\nabla_u \overline{g}] = -\frac{1}{2} \cdot h_u[\overline{g}(\nabla_u g)\overline{g}] = \\ &= -\frac{1}{2} \cdot \overline{g}[\nabla_u g] + \frac{1}{2 \cdot e} \cdot (\nabla_u g)(u,u) \ , \end{aligned} \tag{98}$$

and therefore,

$$\frac{1}{2} \cdot \overline{g}[\nabla_u g] = [\frac{1}{2 \cdot e} \cdot (\nabla_u g)(u,u) - {}_{\nabla}\theta] \quad . \tag{99}$$

It follows for the change $\nabla_u(d\omega)$ of the invariant volume element $d\omega$

$$\begin{aligned} \nabla_u(d\omega) &= \frac{1}{2} \cdot \overline{g}[\nabla_u g] \cdot d\omega = \\ &= [-{}_{\nabla}\theta + \frac{1}{2 \cdot e} \cdot (\nabla_u g)(u,u)] \cdot d\omega \quad . \end{aligned} \tag{100}$$

Special case: (Pseudo) Riemannian spaces with or without torsion (U_n or V_n-spaces): metric transport $\nabla_u g := 0$.

$$\nabla_u(d\omega) = -{}_{\nabla}\theta \cdot d\omega = 0 \quad . \tag{101}$$

This means that the invariant volume element $d\omega$ does not change under a transport of $d\omega$ along u in (pseudo) Riemannian spaces with or without torsion. This result follows directly from $\nabla_u(d\omega) = (1/2) \cdot \overline{g}[\nabla_u g] \cdot d\omega$.

In an analogous way, for the expansion velocity ${}_{\pounds}\theta$ induced by the Lie differential operator the relations are valid:

$$\begin{aligned} {}_{\pounds}\theta &= \frac{1}{2} \cdot h_u[\pounds_u \overline{g}] = -\frac{1}{2} \cdot h_u[\overline{g}(\pounds_u g)\overline{g}] = \\ &= -\frac{1}{2} \cdot \overline{g}[\pounds_u g] + \frac{1}{2 \cdot e} \cdot (\pounds_u g)(u,u) \ , \end{aligned} \tag{102}$$

$$\begin{aligned} \pounds_u(d\omega) &= \frac{1}{2} \cdot \overline{g}[\pounds_u g] \cdot d\omega = \\ &= [-{}_{\pounds}\theta + \frac{1}{2 \cdot e} \cdot (\pounds_u g)(u,u)] \cdot d\omega \quad . \end{aligned} \tag{103}$$

If we use further the explicit form of $\pounds_u g$ for spaces with affine connections and metrics

$$\begin{aligned} \pounds_u g &= (\pounds_u g_{ij}) \cdot dx^i.dx^j = \\ &= [g_{ij;k} \cdot u^k + g_{kj} \cdot u^{\overline{k}}{}_{;\underline{i}} + g_{ik} \cdot u^{\overline{k}}{}_{;\underline{j}} + \\ &\quad +(g_{kj} \cdot T_{l\underline{i}}{}^{\overline{k}} + g_{ik} \cdot T_{l\underline{j}}{}^{\overline{k}}) \cdot u^l] \cdot dx^i.dx^j \quad , \end{aligned} \tag{104}$$

and the relation

$$(\pounds_u g)(u,u) = \nabla_u[g(u,u)] = ue \ , \tag{105}$$

we obtain

$$\frac{1}{2}\cdot\overline{g}[\pounds_u g] = (-\,_{\pounds}\theta+\frac{ue}{2\cdot e})\ , \tag{106}$$

$$\pounds_u(d\omega) = \frac{1}{2}\cdot\overline{g}[\pounds_u g]\cdot d\omega=(-\,_{\pounds}\theta+\frac{ue}{2\cdot e})\cdot d\omega\quad . \tag{107}$$

For a normalized vector field u with $e=$const. and therefore, $ue=0$, it follows

$$\pounds_u(d\omega)=-\,_{\pounds}\theta\cdot d\omega\quad . \tag{108}$$

The last expression leads to the interpretation of $_{\pounds}\theta$ as expansion velocity induced by the Lie differential operator $\pounds_u$ on the covariant metric tensor g. In other words, $_{\pounds}\theta$ is the expansion velocity induced by a dragging of $d\omega$ along a vector field u.

Since

$$\begin{aligned}\nabla_u(d\omega)-\pounds_u(d\omega) &= \{\frac{1}{2}\cdot\overline{g}[\nabla_u g]-\frac{1}{2}\cdot\overline{g}[\pounds_u g]\}\cdot d\omega= \\ &= \{\frac{1}{2}\cdot\overline{g}[\nabla_u g-\pounds_u g]\}\cdot d\omega= \\ &= [-\,_{\nabla}\theta+\frac{1}{2\cdot e}\cdot(\nabla_u g)(u,u)+\,_{\pounds}\theta-\frac{ue}{2\cdot e}]\cdot d\omega= \\ &= \{\,_{\pounds}\theta-\,_{\nabla}\theta+\frac{1}{2\cdot e}\cdot[(\nabla_u g)(u,u)-ue]\}.d\omega= \\ &= \{-\theta+\frac{1}{2\cdot e}\cdot[(\nabla_u g)(u,u)-ue]\}.d\omega \end{aligned} \tag{109}$$

and

$$\begin{aligned}(\nabla_u g)(u,u)-ue &= (\nabla_u g)(u,u)-\nabla_u[g(u,u)]= \\ &= -2\cdot g(u,\nabla_u u)=-2\cdot g(u,a)\quad ,\qquad \nabla_u u=a\ ,\end{aligned} \tag{110}$$

it follows that

$$\nabla_u(d\omega)-\pounds_u(d\omega)=-[\theta+\frac{1}{e}\cdot g(u,a)]\cdot d\omega\quad . \tag{111}$$

If the vector field u is an auto-parallel vector field ($\nabla_u u=a=0$), then

$$\nabla_u(d\omega)-\pounds_u(d\omega)=-\theta\cdot d\omega\quad . \tag{112}$$

Therefore, the expansion velocity θ determines the difference between the change of the invariant volume element $d\omega$ after a transport along an auto-parallel curve and a dragging along the vector field u.

The trace-free symmetric tensor field σ is called shear velocity tensor (shear velocity, shear)

$$\sigma=\,_{\nabla}\sigma-\,_{\pounds}\sigma\quad ,\qquad \overline{g}[\sigma]=0\quad ,\qquad \overline{g}[_{\nabla}\sigma]=0\quad ,\qquad \overline{g}[_{\pounds}\sigma]=0\ . \tag{113}$$

The tensor field $_{\nabla}\sigma$ is called shear velocity induced by a transport along the vector field u, $_{\pounds}\sigma$ is called shear velocity induced by a dragging along u. The change of the length of the vector $\xi_{(a)\perp}$ along a curve $x^i(\tau,\lambda_0)$ is determined only by $_{\pounds}\sigma$ and not by $_{\nabla}\sigma$. The shear velocity tensor $_{\pounds}\sigma$ determines the change of the length of the vector $\xi_{(a)\perp}$ together with the expansion velocity $_{\pounds}\theta$. The shear velocity $_{\pounds}\sigma$ does not contain the part, responding for the change of the invariant volume element. This is because $_{\pounds}\sigma$ is constructed by a tensor, which trace $_{\pounds}\theta$, determining the change of the invariant volume element $d\omega$, is substracted from it. Thus the shear velocity tensor does not change a volume in a space with affine connections and metrics. It generates a volume-preserving shape deformation. This means that a sphere could be deformed to an ellipsoid under keeping its volume if $_{\pounds}\sigma\neq 0$.

We can now resume, that the expansion velocity and the shear velocity have their corresponding physical meaning in the continuum media mechanics in spaces with affine connections and metrics.

The tensor field ω, with $\omega(u)=-(u)(\omega)=0$, is called rotation (vortex) velocity tensor (rotation velocity, rotation, vortex). It does not change the length of a vector field $\xi_{(a)\perp}$ and therefore, it changes only the direction of $\xi_{(a)\perp}$ causing its rotation $[\omega(\xi_{(a)\perp})]$, with $(u)(\omega(\xi_{(a)\perp})=0$, in the $n-1$ dimensional subspace, orthogonal to the vector u.

2.4 Relative velocity and contravariant vector fields

The kinematic characteristics related to the notion of relative velocity can be used in finding out their influence on the rate of change of the length of a contravariant vector field as well as the rate of change of the cosine between two contravariant vector fields.

2.4.1 Relative velocity and change of the length of a contravariant vector field

Let we now consider the influence of the kinematic characteristics related to the relative velocity upon the change of the length of a contravariant vector field.

Let $l_\xi =\mid g(\xi,\xi) \mid^{\frac{1}{2}}$ be the length of a contravariant vector field ξ. The rate of change ul_ξ of l_ξ along a contravariant vector field u can be expressed in the form $\pm\, 2 \cdot l_\xi \cdot (ul_\xi) = (\nabla_u g)(\xi,\xi) + 2 \cdot g(\nabla_u \xi, \xi)$. By the use of the projections of ξ and $\nabla_u \xi$ along and orthogonal to u (see the section about kinematic characteristics and relative velocity) we can find the relations

$$\begin{gathered} 2 \cdot g(\nabla_u \xi, \xi) = 2 \cdot \tfrac{l}{e} \cdot g(\nabla_u \xi, u) + 2 \cdot g({}_{rel}v, \xi_\perp) \ , \\ (\nabla_u g)(\xi,\xi) = (\nabla_u g)(\xi_\perp,\xi_\perp) + 2 \cdot \tfrac{l}{e} \cdot (\nabla_u g)(\xi_\perp, u) + \tfrac{l^2}{e^2} \cdot (\nabla_u g)(u,u) \ . \end{gathered}$$

Then, it follows for $\pm\, 2 \cdot l_\xi \cdot (ul_\xi)$ the expression

$$\begin{gathered} \pm\, 2 \cdot l_\xi \cdot (ul_\xi) = (\nabla_u g)(\xi_\perp,\xi_\perp) + 2 \cdot \tfrac{l}{e} \cdot [(\nabla_u g)(\xi_\perp, u) + g(\nabla_u \xi, u)] + \\ + \tfrac{l^2}{e^2} \cdot (\nabla_u g)(u,u) + 2 \cdot g({}_{rel}v, \xi_\perp) \ , \end{gathered} \tag{114}$$

where

$$g({}_{rel}v, \xi_\perp) = \frac{l}{e} \cdot h_u(a, \xi_\perp) + h_u(\pounds_u \xi, \xi_\perp) + d(\xi_\perp, \xi_\perp) \ , \tag{115}$$

$$d(\xi_\perp, \xi_\perp) = \sigma(\xi_\perp, \xi_\perp) + \frac{1}{n-1} \cdot \theta \cdot l^2_{\xi_\perp} \ . \tag{116}$$

For finding out the last two expressions the following relations have been used:

$$g(\overline{g}(h_u)a, \xi_\perp) = h_u(a, \xi_\perp) \ , \quad g(\overline{g}(h_u)(\pounds_u \xi), \xi_\perp) = h_u(\pounds_u \xi, \xi_\perp) \ , \tag{117}$$

$$g(\overline{g}[d(\xi)], \xi_\perp) = d(\xi_\perp, \xi_\perp) \ , \quad d(\xi) = d(\xi_\perp) \ . \tag{118}$$

Special case: $g(u, \xi) = l := 0 : \xi = \xi_\perp$.

$$\pm\, 2 \cdot l_{\xi_\perp} \cdot (ul_{\xi_\perp}) = (\nabla_u g)(\xi_\perp,\xi_\perp) + 2 \cdot g({}_{rel}v, \xi_\perp) \ . \tag{119}$$

Special case: V_n-spaces: $\nabla_\eta g = 0$ for $\forall \eta \in T(M)$ ($g_{ij;k} = 0$), $g(u,\xi) = l := 0 : \xi = \xi_\perp$.

$$\pm\, l_{\xi_\perp} \cdot (ul_{\xi_\perp}) = g({}_{rel}v, \xi_\perp) \ . \tag{120}$$

In $(\overline{L}_n, g)$-spaces as well as in (L_n, g)-spaces the covariant derivative $\nabla_u g$ of the metric tensor field g along u can be decomposed in its trace free part ${}^s\nabla_u g$ and its trace part $\frac{1}{n} \cdot Q_u \cdot g$ as

$$\nabla_u g = {}^s\nabla_u g + \frac{1}{n} \cdot Q_u \cdot g \ , \qquad \dim M = n \ ,$$

where

$$\overline{g}[{}^s\nabla_u g] = 0 \ , \quad Q_u = \overline{g}[\nabla_u g] = g^{\overline{kl}} \cdot g_{kl;j} \cdot u^j = Q_j \cdot u^j \ , \quad Q_j = g^{\overline{kl}} \cdot g_{kl;j} \ .$$

The covariant vector $\overline{Q} = \frac{1}{n} \cdot Q = \frac{1}{n} \cdot Q_j \cdot dx^j = \frac{1}{n} \cdot Q_\alpha \cdot e^\alpha$ is called *Weyl's covariant vector field*. The operator $\nabla_u = {}^s\nabla_u + \frac{1}{n} \cdot Q_u$ is called *trace free covariant operator*.

If we use now the decomposition of $\nabla_u g$ in the expression for $\pm\, 2 \cdot l_\xi \cdot (ul_\xi)$ we find the relation

$$\begin{gathered} \pm\, 2 \cdot l_\xi \cdot (ul_\xi) = ({}^s\nabla_u g)(\xi,\xi) \pm \tfrac{1}{n} \cdot Q_u \cdot l^2_\xi + 2 \cdot g(\nabla_u \xi, \xi) = \\ = ({}^s\nabla_u g)(\xi_\perp,\xi_\perp) + \\ + \tfrac{l}{e} \cdot [2 \cdot ({}^s\nabla_u g)(\xi_\perp, u) + 2 \cdot g(\nabla_u \xi, u) + \tfrac{l}{e} \cdot ({}^s\nabla_u g)(u,u)] + \\ + \tfrac{1}{n} \cdot Q_u \cdot (\pm\, l^2_{\xi_\perp} + \tfrac{l^2}{e}) + 2 \cdot g({}_{rel}v, \xi_\perp) \ , \end{gathered} \tag{121}$$

where $\pm\, l^2_{\xi_\perp} = g(\xi_\perp, \xi_\perp)$, $l = g(\xi, u)$.

For $l_\xi \neq 0$:

$$ul_\xi = \pm\frac{1}{2\cdot l_\xi}\cdot({}^s\nabla_u g)(\xi,\xi) + \frac{1}{2\cdot n}\cdot Q_u\cdot l_\xi \pm \frac{1}{l_\xi}\cdot g(\nabla_u\xi,\xi) \ . \tag{122}$$

In the case of a parallel transport ($\nabla_u\xi = 0$) of ξ along u the change ul_ξ of the length l_ξ is

$$ul_\xi = \pm\frac{1}{2.l_\xi}\cdot({}^s\nabla_u g)(\xi,\xi) + \frac{1}{2\cdot n}\cdot Q_u\cdot l_\xi \ . \tag{123}$$

Special case: $\nabla_u\xi = 0$ and ${}^s\nabla_u g = 0$.

$$ul_\xi = \frac{1}{2\cdot n}\cdot Q_u\cdot l_\xi \ . \tag{124}$$

If $u = \frac{d}{ds} = u^i\cdot\partial_i = (dx^i/ds)\cdot\partial_i$, then

$$\begin{aligned}
l_\xi(s+ds) &\approx l_\xi(s) + \frac{dl_\xi}{ds}\cdot ds = l_\xi(s) + \frac{1}{2\cdot n}\cdot Q_u(s)\cdot l_\xi(s)\cdot ds = \\
&= (1+\frac{1}{2\cdot n}\cdot Q_u(s)\cdot ds)\cdot l_\xi(s) = \triangle_u(s)\cdot l_\xi(s) \ , \\
\triangle_u(s) &= 1+\frac{1}{2\cdot n}\cdot Q_u(s)\cdot ds \ , \qquad (125)\\
\frac{dl_\xi}{ds} &= \lim_{ds\to 0}\frac{l_\xi(s+ds)-l_\xi(s)}{ds} = +\frac{1}{2\cdot n}\cdot Q_u(s)\cdot l_\xi(s) \qquad (126)
\end{aligned}$$

Therefore, the rate of change of l_ξ along u is linear to l_ξ.
Special case: $g(u,\xi) = l := 0 : \xi = \xi_\perp$.

$$\pm 2\cdot l_{\xi_\perp}\cdot(ul_{\xi_\perp}) = ({}^s\nabla_u g)(\xi_\perp,\xi_\perp) \pm \frac{1}{n}\cdot Q_u\cdot l^2_{\xi_\perp} + 2\cdot g({}_{rel}v,\xi_\perp) \ .$$

$$ul_{\xi_\perp} = \pm\frac{1}{2\cdot l_{\xi_\perp}}\cdot({}^s\nabla_u g)(\xi_\perp,\xi_\perp) + \frac{1}{2\cdot n}\cdot Q_u\cdot l_{\xi_\perp} \pm \frac{1}{l_{\xi_\perp}}\cdot g({}_{rel}v,\xi_\perp) \ , \qquad l_{\xi_\perp}\neq 0 \ . \tag{127}$$

Special case: Quasi-metric transports: $\nabla_u g := 2\cdot g(u,\eta)\cdot g$, $\ u,\eta\in T(M)$.

$$\pm 2\cdot l_\xi\cdot(ul_\xi) = 2\cdot g(u,\eta)\cdot(\pm l^2_{\xi_\perp} + \frac{l^2}{e}) + 2\cdot[\frac{l}{e}\cdot g(\nabla_u\xi,u) + g({}_{rel}v,\xi_\perp)] \ . \tag{128}$$

2.4.2 Relative velocity and change of the cosine between two contravariant vector fields

The cosine between two contravariant vector fields ξ and η has been defined as $g(\xi,\eta) = l_\xi\cdot l_\eta\cdot\cos(\xi,\eta)$. The rate of change of the cosine along a contravariant vector field u can be found in the form

$$\begin{aligned}
l_\xi\cdot l_\eta\cdot\{u[\cos(\xi,\eta)]\} = (\nabla_u g)(\xi,\eta) + g(\nabla_u\xi,\eta) + g(\xi,\nabla_u\eta) - \\
-[l_\eta\cdot(ul_\xi) + l_\xi\cdot(ul_\eta)]\cdot\cos(\xi,\eta) \ .
\end{aligned} \tag{129}$$

Special case: $\nabla_u\xi = 0$, $\nabla_u\eta = 0$, ${}^s\nabla_u g = 0$.

$$l_\xi\cdot l_\eta\cdot\{u[\cos(\xi,\eta)]\} = \frac{1}{n}\cdot Q_u\cdot g(\xi,\eta) - [l_\eta\cdot(ul_\xi) + l_\xi\cdot(ul_\eta)]\cdot\cos(\xi,\eta) \ .$$

Since $g(\xi,\eta) = l_\xi\cdot l_\eta\cdot\cos(\xi,\eta)$, it follows from the last relation

$$l_\xi\cdot l_\eta\cdot\{u[\cos(\xi,\eta)]\} = \{\frac{1}{n}\cdot Q_u\cdot l_\xi\cdot l_\eta - [l_\eta\cdot(ul_\xi) + l_\xi\cdot(ul_\eta)]\}\cdot\cos(\xi,\eta) \ .$$

Therefore, if $\cos(\xi,\eta) = 0$ between two parallel transported along u vector fields ξ and η, then the right angle between them [determined by the condition $\cos(\xi,\eta) = 0$] does not change along the contravariant vector field u. In the cases, when $\cos(\xi,\eta) \neq 0$, the rate of change of the cosine of the angle between two vector fields ξ and η is linear to $\cos(\xi,\eta)$.

By the use of the definitions and the relations:

$${}_{rel}v_\xi := \overline{g}[h_u(\nabla_u\xi)] = {}_{rel}v \ , \qquad {}_{rel}v_\eta := \overline{g}[h_u(\nabla_u\eta)] \ , \tag{130}$$

$$\begin{array}{c} g(\nabla_u\xi,\eta) = \frac{1}{e}\cdot g(u,\eta)\cdot g(\nabla_u\xi,u) + g({}_{rel}v_\xi,\eta) \ , \\ g(\nabla_u\eta,\xi) = \frac{1}{e}\cdot g(u,\xi)\cdot g(\nabla_u\eta,u) + g({}_{rel}v_\eta,\xi) \ , \end{array} \tag{131}$$

$$(\nabla_u g)(\xi,\eta) = ({}^s\nabla_u g)(\xi,\eta) + \frac{1}{n}\cdot Q_u\cdot g(\xi,\eta) \ , \tag{132}$$

$$\begin{array}{c} ({}^s\nabla_u g)(\xi,\eta) = ({}^s\nabla_u g)(\xi_\perp,\eta_\perp) + \frac{l}{e}\cdot({}^s\nabla_u g)(u,\eta_\perp) + \frac{\overline{l}}{e}\cdot({}^s\nabla_u g)(\xi_\perp,u)+ \\ +\frac{l}{e}\cdot\frac{\overline{l}}{e}\cdot({}^s\nabla_u g)(u,u) \ , \quad \overline{l} = g(u,\eta) \ , \quad \eta_\perp = \overline{g}[h_u(\eta)] \ , \quad l = g(u,\xi) \ , \end{array} \tag{133}$$

$$\begin{array}{c} (\nabla_u g)(\xi,\eta) = ({}^s\nabla_u g)(\xi,\eta) + \frac{1}{n}\cdot Q_u\cdot g(\xi,\eta) = \\ = ({}^s\nabla_u g)(\xi_\perp,\eta_\perp) + \frac{l}{e}\cdot({}^s\nabla_u g)(u,\eta_\perp) + \frac{\overline{l}}{e}\cdot({}^s\nabla_u g)(\xi_\perp,u)+ \\ +\frac{l}{e}\cdot\frac{\overline{l}}{e}\cdot({}^s\nabla_u g)(u,u) + \frac{1}{n}\cdot Q_u\cdot[\frac{l\cdot\overline{l}}{e} + g(\xi_\perp,\eta_\perp)] \ , \end{array} \tag{134}$$

the expression of $l_\xi\cdot l_\eta\cdot\{u[\cos(\xi,\eta)]\}$ follows in the form

$$\begin{array}{c} l_\xi\cdot l_\eta\cdot\{u[\cos(\xi,\eta)]\} = ({}^s\nabla_u g)(\xi_\perp,\eta_\perp) + \frac{l}{e}\cdot[({}^s\nabla_u g)(u,\eta_\perp) + g(\nabla_u\eta,u)]+ \\ +\frac{\overline{l}}{e}\cdot[({}^s\nabla_u g)(\xi_\perp,u) + g(\nabla_u\xi,u)] + \frac{l\cdot\overline{l}}{e^2}\cdot({}^s\nabla_u g)(u,u)+ \\ +\frac{1}{n}\cdot Q_u\cdot[\frac{l\cdot\overline{l}}{e} + g(\xi_\perp,\eta_\perp)] + g({}_{rel}v_\xi,\eta) + g({}_{rel}v_\eta,\xi)- \\ -[l_\eta\cdot(ul_\xi) + l_\xi\cdot(ul_\eta)]\cdot\cos(\xi,\eta) \ . \end{array} \tag{135}$$

Special case: $g(u,\xi) = l := 0$, $g(u,\eta) = \overline{l} := 0 : \xi = \xi_\perp$, $\eta = \eta_\perp$.

$$\begin{array}{c} l_{\xi_\perp}\cdot l_{\eta_\perp}\cdot\{u[\cos(\xi_\perp,\eta_\perp)]\} = ({}^s\nabla_u g)(\xi_\perp,\eta_\perp) + \frac{1}{n}\cdot Q_u\cdot l_{\xi_\perp}\cdot l_{\eta_\perp}\cdot\cos(\xi_\perp,\eta_\perp)+ \\ +g({}_{rel}v_{\xi_\perp},\eta_\perp) + g({}_{rel}v_{\eta_\perp},\xi_\perp) - [l_{\eta_\perp}\cdot(ul_{\xi_\perp}) + l_{\xi_\perp}\cdot(ul_{\eta_\perp})]\cdot\cos(\xi_\perp,\eta_\perp) \ , \end{array} \tag{136}$$

where $g(\xi_\perp,\eta_\perp) = l_{\xi_\perp}\cdot l_{\eta_\perp}\cdot\cos(\xi_\perp,\eta_\perp)$.

The kinematic characteristics related to the relative velocity and used in considerations of the rate of change of the length of a contravariant vector field as well as the change of the angle between two contravariant vector fields could also be useful for description of the motion of physical systems in $(\overline{L}_n, g)$-spaces.

2.5 Expansion velocity and variation of the invariant volume element

From the explicit form of the expansion velocity θ

$$\begin{array}{c} \theta = \frac{1}{2}\cdot h_u[\nabla_u\overline{g} - \pounds_u\overline{g}] = \\ = \frac{1}{2}\cdot\{\overline{g}[\pounds_u g] - \overline{g}[\nabla_u g] + \frac{1}{e}\cdot(\nabla_u g)(u,u) - \frac{1}{e}\cdot(\pounds_u g)(u,u)\} \ , \end{array} \tag{137}$$

where

$$[g(u)\otimes g(u)][\nabla_u\overline{g}] = -(\nabla_u g)(u,u) \ , \tag{138}$$

$$[g(u)\otimes g(u)][\pounds_u\overline{g}] = -(\pounds_u g)(u,u) \ , \tag{139}$$

one can draw the conclusion that the variation of the invariant volume element $d\omega$ [6] is connected with the expansion velocity θ. From

$$\nabla_u(d\omega) = \frac{1}{2}\cdot\overline{g}[\nabla_u g]\cdot d\omega \ , \qquad \pounds_\xi(d\omega) = \frac{1}{2}\cdot\overline{g}[\pounds_u g]\cdot d\omega$$

and (137) the following relations are fulfilled

$$\theta\cdot d\omega = \pounds_u(d\omega) - \nabla_u(d\omega) + \frac{1}{2\cdot e}\cdot[(\nabla_u g)(u,u) - (\pounds_u g)(u,u)]\cdot d\omega \ , \tag{140}$$

$$\begin{array}{c} \pounds_u(d\omega) - \nabla_u(d\omega) = \frac{1}{2}\cdot\overline{g}[\pounds_u g - \nabla_u g]\cdot d\omega = \\ = [\theta + \frac{1}{2e}\cdot(\pounds_u g - \nabla_u g)(u,u)]\cdot d\omega \ , \end{array} \tag{141}$$

$$\pounds_u(d\omega) = [\theta + \frac{1}{2}\cdot\overline{g}[\nabla_u g] + \frac{1}{2\cdot e}\cdot(\pounds_u g - \nabla_u g)(u,u)]\cdot d\omega \ , \tag{142}$$

$$\nabla_u(d\omega) = [-\theta + \frac{1}{2}\cdot\overline{g}[\pounds_u g] + \frac{1}{2\cdot e}\cdot(\nabla_u g - \pounds_u g)(u,u)]\cdot d\omega \ , \tag{143}$$

where

$$\frac{1}{2}\cdot g[\nabla_u\overline{g}] = \frac{1}{2}\cdot h_u[\nabla_u\overline{g}] - \frac{1}{2\cdot e}\cdot(\nabla_u g)(u,u) = \tag{144}$$

$$= \theta + \frac{1}{2} \cdot h_u[\pounds_u \overline{g}] - \frac{1}{2 \cdot e} \cdot (\nabla_u g)(u, u) \ , \tag{145}$$

$$\begin{aligned} \nabla_u(d\omega) &= \tfrac{1}{2} \cdot \overline{g}[\nabla_u g] \cdot d\omega = -\tfrac{1}{2} \cdot g[\nabla_u \overline{g}] \cdot d\omega = \\ &= \{\tfrac{1}{2 \cdot e} \cdot (\nabla_u g)(u, u) - \tfrac{1}{2} \cdot h_u[\nabla_u \overline{g}]\} \cdot d\omega = \\ &= \{-\theta - \tfrac{1}{2} \cdot h_u[\pounds_u \overline{g}] + \tfrac{1}{2 \cdot e} \cdot (\nabla_u g)(u, u)\} \cdot d\omega \ . \end{aligned} \tag{146}$$

Special case: Metric transports ($\nabla_u g = 0$):

$$\theta = -\frac{1}{2} \cdot h_u[\pounds_u \overline{g}] = \frac{1}{2} \cdot \{\overline{g}[\pounds_u g] - \frac{1}{e} \cdot (\pounds_u g)(u, u)\}, \tag{147}$$

$$\nabla_u(d\omega) = 0, \tag{148}$$

$$\pounds_u(d\omega) = [\theta + \frac{1}{2 \cdot e} \cdot (\pounds_u g)(u, u)] \cdot d\omega \ . \tag{149}$$

If the additional condition $g(u, u) = e =$const. is fulfilled, then $\pounds_u[g(u, u)] = 0 = (\pounds_u g)(u, u)$ and

$$\pounds_u(d\omega) = \theta \cdot d\omega \ .$$

Special case: Metric transports ($\nabla_u g = 0$) and isometric draggings-along (motions) ($\pounds_u g = 0$):

$$\theta = 0 \ , \ \nabla_u(d\omega) = 0 \ , \ \pounds_u(d\omega) = 0 \ . \tag{150}$$

At the same time,

$$\begin{aligned} \pounds_u(d\omega) &= \tfrac{1}{2} \cdot \overline{g}[\pounds_u g] \cdot d\omega = -\tfrac{1}{2} \cdot g[\pounds_u \overline{g}] \cdot d\omega = \\ &= \{\tfrac{1}{2 \cdot e} \cdot (\pounds_u g)(u, u) - \tfrac{1}{2} \cdot h_u[\pounds_u \overline{g}]\} \cdot d\omega = \\ &= \{\theta - \tfrac{1}{2} \cdot h_u[\nabla_u \overline{g}] + \tfrac{1}{2 \cdot e} \cdot (\pounds_u g)(u, u)\} \cdot d\omega \ . \end{aligned} \tag{151}$$

After introducing the abbreviations

$${}_{l}\theta_u = \frac{1}{2} \cdot \overline{g}[\pounds_u g] \ , \qquad {}_{c}\theta_u = \frac{1}{2} \cdot \overline{g}[\nabla_u g] \ , \tag{152}$$

$${}_{r}\theta_u = \frac{1}{2 \cdot e} \cdot (\pounds_u g - \nabla_u g)(u, u) \ , \tag{153}$$

θ, $\nabla_u(d\omega)$, $\pounds_u(d\omega)$ and $\pounds_u g - \nabla_u g$ can be written in the form

$$\theta = {}_{l}\theta_u - {}_{c}\theta_u - {}_{r}\theta_u \ , \tag{154}$$

$$\nabla_u(d\omega) = {}_{c}\theta_u \cdot d\omega = (-\theta + {}_{l}\theta_u - {}_{r}\theta_u) \cdot d\omega \ , \tag{155}$$

$$\pounds_u(d\omega) = {}_{l}\theta_u \cdot d\omega = (\theta + {}_{c}\theta_u + {}_{r}\theta_u) \cdot d\omega \ , \tag{156}$$

$$\pounds_u(d\omega) - \nabla_u(d\omega) = ({}_{l}\theta_u - {}_{c}\theta_u) \cdot d\omega \ . \tag{157}$$

The variation of the invariant volume element along a contravariant vector field, orthogonal to the contravariant vector field u can be found by means of the projections of a contravariant vector field ξ along u

$$\xi = \frac{l}{e} \cdot u + \xi_{\perp} \ , \qquad \nabla_{\xi} = \frac{l}{e} \cdot \nabla_u + \nabla_{\xi_{\perp}} \ ,$$

$$\nabla_{\xi}(d\omega) = \frac{l}{e} \cdot \nabla_u(d\omega) + \nabla_{\xi_{\perp}}(d\omega) \ , \tag{158}$$

$$\nabla_{\xi_{\perp}}(d\omega) = \frac{1}{2} \cdot \overline{g}[\nabla_{\xi_{\perp}} g] \cdot d\omega \ . \tag{159}$$

By means of the relations (146) and (151) the following propositions can be proved:

Proposition 2 *The necessary and sufficient condition for the existence of the covariant derivative $\nabla_u(d\omega)$ of the invariant volume element $d\omega$ in the form*

$$\nabla_u(d\omega) = -\theta \cdot d\omega \tag{160}$$

is the condition

$$h_u[\pounds_u \overline{g}] = \frac{1}{e} \cdot (\nabla_u g)(u, u) \ . \tag{161}$$

Proof: 1. Sufficiency. From (161) and (146)

$$\nabla_u(d\omega) = \{-\theta - \frac{1}{2} \cdot h_u[\pounds_u \overline{g}] + \frac{1}{2e} \cdot (\nabla_u g)(u,u)\} \cdot d\omega \ ,$$

it follows

$$\nabla_u(d\omega) = -\theta \cdot d\omega \ .$$

2. Necessity. From (160) and (146), it follows

$$\{-h_u[\pounds_u \overline{g}] + \frac{1}{e} \cdot (\nabla_u g)(u,u)\} \cdot d\omega = 0 \ ,$$

from where, for $d\omega \neq 0$, (161) follows.

Proposition 3 *The necessary and sufficient condition for the existence of the Lie derivative $\pounds_u(d\omega)$ of the invariant volume element $d\omega$ in the form*

$$\pounds_u(d\omega) = \theta \cdot d\omega \tag{162}$$

is the condition

$$h_u[\nabla_u \overline{g}] = \frac{1}{e} \cdot (\pounds_u g)(u,u) \ . \tag{163}$$

Proof: 1. Sufficiency. From (163) and (151)

$$\pounds_u(d\omega) = \{\theta - \frac{1}{2} \cdot h_u[\nabla_u \overline{g}] + \frac{1}{2 \cdot e} \cdot (\pounds_u g)(u,u)\} \cdot d\omega$$

it follows

$$\pounds_u(d\omega) = \theta \cdot d\omega \ .$$

2. Necessity. From (162) and (151), it follows

$$\{-h_u[\nabla_u \overline{g}] + \frac{1}{e} \cdot (\pounds_u g)(u,u)\} \cdot d\omega = 0 \ ,$$

from where, for $d\omega \neq 0$, (163) follows.

Special case: Quasi-projective non-metric transports

$$\nabla_u g = \tfrac{1}{2} \cdot [p \otimes g(u) + g(u) \otimes p] \ ,$$
$$\nabla_u \overline{g} = \tfrac{1}{2} \cdot (v \otimes u + u \otimes v) \ , \ v = -\overline{g}(p) \ ,$$

$$h_u[\nabla_u \overline{g}] = 0 \ , \tag{164}$$

$$(\nabla_u g)(u,u) = e \cdot p(u) \ , \qquad e = g(u,u) \neq 0 \ , \tag{165}$$

$$\nabla_u(d\omega) = \frac{1}{2} \cdot p(u) \cdot d\omega = \frac{1}{2 \cdot e} \cdot (\nabla_u g)(u,u) \cdot d\omega \ , \tag{166}$$

$$\pounds_u(d\omega) = [\theta + \frac{1}{2 \cdot e} \cdot (\pounds_u g)(u,u)] \cdot d\omega \ . \tag{167}$$

The variation of the invariant volume element and its preservation is connected with the structures of a Lagrangian theory of tensor fields over $(\overline{L}_n, g)$-spaces [**?**].

2.6 Rotation (vortex) velocity

The tensor ω is called rotation (vortex) velocity tensor (rotation velocity, vortex velocity, rotation, vortex). It does not change the length of a vector field $\xi_\perp$ and, therefore, ω changes only the direction of $\xi_\perp$, causing its rotation in the $n-1$ dimensional sub space, orthogonal to u [because of the relation $[\omega(\xi_\perp)](u) = (u)(\omega)(\xi_\perp) = \omega(u, \xi_\perp) = 0$]. By the use of the Levi-Civita symbols or the star operator $*$ we can define the corresponding rotation velocity vector.

2.6.1 Definition of the rotation (vortex) velocity vector

For a differentiable manifold M with $\dim M = 4$, a vector $\overline{\omega}$ corresponding to the rotation velocity tensor ω could be defined by the use of ω, u, and the Hodge (star) operator $*$

$$\overline{\omega} := \overline{g}(*\, (g(u) \wedge \omega)) \ , \qquad \overline{\omega} = \overline{\omega}^i \cdot \partial_i \quad , \quad \overline{\omega} \in T(M) \ . \tag{168}$$

Let us find now the explicitly form of the rotation (vortex) velocity vector $\overline{\omega}$. For this purpose, we should write $\omega, g(u)$, and $\overline{g}$ in a co-ordinate (or non-co-ordinate) basis

$$\overline{g} = g^{ij} \cdot \partial_i.\partial_j \ , \quad g(u) = g_{i\overline{k}} \cdot u^k \cdot dx^i \ , \quad \omega = \omega_{ij} \cdot dx^i \wedge dx^j \ . \tag{169}$$

Then $g(u) \wedge \omega$ will have the form

$$\begin{aligned} g(u) \wedge \omega &= g_{im} \cdot u^{\overline{m}} \cdot \omega_{jk} \cdot dx^i \wedge dx^j \wedge dx^k = \\ &= {}_aA_{[ijk]} \cdot dx^i \wedge dx^j \wedge dx^k = {}_aA \ , \\ {}_aA &: = g(u) \wedge \omega \ , \end{aligned} \tag{170}$$

where

$$ {}_aA_{[ijk]} = u^{\overline{m}} \cdot g_{m[i}\omega_{jk]} \ , \tag{171}$$

$$\begin{aligned} {}_a\overline{A} &= A^{[ijk]} \cdot \partial_i \wedge \partial_j \wedge \partial_k \ , \\ A^{[ijk]} &= g^{i\overline{l}} \cdot g^{j\overline{m}} \cdot g^{k\overline{n}} \cdot {}_aA_{[lmn]} = \\ &= g^{i\overline{l}} \cdot g^{j\overline{m}} \cdot g^{k\overline{n}} \cdot u^{\overline{r}} \cdot g_{r[l}\omega_{mn]} \ . \end{aligned} \tag{172}$$

For $\dim M = 4$, $k = 3$, we have

$$*\, ({}_aA) = [*\, ({}_aA)]_s \cdot dx^s = \frac{1}{3!} \cdot \sqrt{-d_g} \cdot \varepsilon_{sijk} \cdot g^{i\overline{l}} \cdot g^{\overline{j}\overline{m}} \cdot g^{\overline{k}\overline{n}} \cdot u^{\overline{r}} \cdot g_{r[l}\omega_{mn]} \cdot dx^s \ , \tag{173}$$

$$\begin{aligned} g_{r[l}\omega_{mn]} &= \frac{1}{3!} \cdot (g_{rl} \cdot \omega_{mn} + g_{rn} \cdot \omega_{ml} + g_{rn} \cdot \omega_{lm} + g_{rm} \cdot \omega_{nl} - g_{rl} \cdot \omega_{nm} - g_{rm} \cdot \omega_{nl}) = \\ &= \frac{1}{3!} \cdot 2 \cdot g_{rl} \cdot \omega_{mn} = \frac{1}{3} \cdot g_{rl} \cdot \omega_{mn} \quad , \qquad \omega_{lm} = -\omega_{ml} \ , \end{aligned} \tag{174}$$

$$\begin{aligned} g^{i\overline{l}} \cdot g^{\overline{j}\overline{m}} \cdot g^{\overline{k}\overline{n}} \cdot u^{\overline{r}} \cdot g_{r[l}\omega_{mn]} &= \frac{1}{3} \cdot g^{i\overline{l}} \cdot g_{rl} \cdot g^{\overline{j}\overline{m}} \cdot g^{\overline{k}\overline{n}} \cdot u^{\overline{r}} \cdot \omega_{mn} = \\ &= \frac{1}{3} \cdot g^i_r \cdot u^{\overline{r}} \cdot g^{\overline{j}\overline{m}} \cdot g^{\overline{k}\overline{n}} \cdot \omega_{mn} = \\ &= \frac{1}{3} \cdot u^{\overline{i}} \cdot \omega^{\overline{j}\overline{k}} \ , \end{aligned} \tag{175}$$

$$\begin{aligned} *\, (g(u) \wedge \omega) &= *\, ({}_aA) = \frac{1}{3!} \cdot \sqrt{-d_g} \cdot \varepsilon_{sijk} \cdot \frac{1}{3} \cdot u^{\overline{i}} \cdot \omega^{\overline{j}\overline{k}} \cdot dx^s = \\ &= \frac{1}{18} \cdot \sqrt{-d_g} \cdot \varepsilon_{ijkl} \cdot u^{\overline{j}} \cdot \omega^{\overline{k}\overline{l}} \cdot dx^i = \overline{\omega}_i \cdot dx^i \ , \end{aligned} \tag{176}$$

$$\overline{\omega}_i = \frac{1}{18} \cdot \sqrt{-d_g} \cdot \varepsilon_{ijkl} \cdot u^{\overline{j}} \cdot \omega^{\overline{k}\overline{l}} \ , \tag{177}$$

$$\overline{\omega} = \overline{g}(*\, (g(u) \wedge \omega)) = g^{i\overline{m}} \cdot \overline{\omega}_m \cdot \partial_i = \overline{\omega}^i \cdot \partial_i \ , \tag{178}$$

$$\overline{\omega}^i = g^{i\overline{m}} \cdot \overline{\omega}_m \cdot = \frac{1}{18} \cdot \sqrt{-d_g} \cdot \varepsilon_{mjkl} \cdot g^{i\overline{m}} \cdot u^{\overline{j}} \cdot \omega^{\overline{k}\overline{l}} \ .$$

All further results for $\overline{\omega}$ are specialized for $\dim M = 4$.

2.6.2 Properties of the rotation (vortex) velocity vector

By the use of the above expressions, we can find some important relations and properties of the rotation (vortex) vector.

1. The rotation (vortex) velocity vector $\overline{\omega}$ is orthogonal to the vector u

$$g(u,\overline{\omega}) = g_{\overline{i}\overline{j}} \cdot u^i \cdot \overline{\omega}^j = 0 \quad . \tag{179}$$

Proof:

$$\begin{aligned} g(u,\overline{\omega}) &= g_{\overline{i}\overline{j}} \cdot u^i \cdot \overline{\omega}^j = g_{\overline{i}\overline{j}} \cdot u^i \cdot \frac{1}{18} \cdot \sqrt{-d_g} \cdot \varepsilon_{mnkl} \cdot g^{j\overline{m}} \cdot u^{\overline{n}} \cdot \omega^{\overline{kl}} = \\ &= \frac{1}{18} \cdot \sqrt{-d_g} \cdot g_{ij} \cdot g^{\overline{j}\overline{m}} \cdot u^{\overline{i}} \cdot u^{\overline{n}} \cdot \varepsilon_{mnkl} \cdot \omega^{\overline{kl}} = \\ &= \frac{1}{18} \cdot \sqrt{-d_g} \cdot g_i^m \cdot \varepsilon_{mnkl} \cdot u^{\overline{i}} \cdot u^{\overline{n}} \cdot \omega^{\overline{kl}} = \\ &= \frac{1}{18} \cdot \sqrt{-d_g} \cdot \varepsilon_{mnkl} \cdot u^{\overline{m}} \cdot u^{\overline{n}} \cdot \omega^{\overline{kl}} = 0 \ . \end{aligned} \tag{180}$$

2. Representation of the rotation velocity tensor ω by means of the rotation velocity vector $\overline{\omega}$.

We can use the definition of the rotation velocity vector $\overline{\omega}$ to express the rotation velocity tensor ω. From $\overline{\omega} = \overline{g}(*(g(u) \wedge \omega))$ we obtain

$$g(\overline{\omega}) = *(g(u) \wedge \omega) \quad , \qquad *\ (g(\overline{\omega})) = *(*(g(u) \wedge \omega)) \ , \tag{181}$$

$$\begin{aligned} *(g(\overline{\omega})) &= \varepsilon \cdot (-1)^{3\cdot(4-3)} \cdot \frac{4!}{(4-3)!3!} \cdot g(u) \wedge \omega = \\ &= \varepsilon \cdot (-1) \cdot 4 \cdot g(u) \wedge \omega \ = \\ &= -4 \cdot \varepsilon \cdot g(u) \wedge \omega \quad . \end{aligned} \tag{182}$$

On the other side the following relations are valid

(a) $S(u, g(u)) = g(u,u) = e$.

Proof:

$$S(u, g(u) \wedge \omega) = S(u, g(u)) \wedge \omega + (-1)^1 \cdot g(u) \cdot S(u,\omega) \ , \tag{183}$$

$$\begin{aligned} S(u, g(u)) &= S(u^i \cdot \partial_i,\ g_{kl} \cdot u^{\overline{l}} \cdot dx^k) = \\ &= u^i \cdot g_{kl} \cdot u^{\overline{l}} \cdot S(\partial_i, dx^k) = u^i \cdot g_{kl} \cdot u^{\overline{l}} \cdot f^k{}_i = \\ &= g_{kl} \cdot u^{\overline{k}} \cdot u^{\overline{l}} = g(u,u) = e \ , \end{aligned} \tag{184}$$

(b) $S(u,\omega) = -\omega(u) = (u)(\omega) = 0$.

Proof:

$$\begin{aligned} S(u,\omega) &= S(u^i \cdot \partial_i, \omega_{kl} \cdot dx^k \wedge dx^l) = \\ &= u^i \cdot \omega_{kl} \cdot \frac{1}{2} \cdot S(\partial_i, dx^k \otimes dx^l - dx^l \otimes dx^k) = \\ &= \frac{1}{2} \cdot u^i \cdot \omega_{kl} \cdot (f^k{}_i \cdot dx^l - f^l{}_i \cdot dx^k) = \\ &= \frac{1}{2} \cdot u^i \cdot (\omega_{kl} \cdot f^k{}_i \cdot dx^l - \omega_{kl} \cdot f^l{}_i \cdot dx^k) = \\ &= \frac{1}{2} \cdot u^i \cdot (\omega_{lk} \cdot f^l{}_i \cdot dx^k - \omega_{kl} \cdot f^l{}_i \cdot dx^k) = \\ &= \frac{1}{2} \cdot u^i \cdot f^l{}_i \cdot (\omega_{lk} - \omega_{kl}) \cdot dx^k = -u^i \cdot f^l{}_i \cdot \omega_{kl} \cdot dx^k = \\ &= -\omega_{kl} \cdot u^{\overline{l}} \cdot dx^k = -\omega(u) = (u)(\omega) = 0 \ , \end{aligned} \tag{185}$$

(c) $S(u, g(u) \wedge \omega) = e \cdot \omega$.

(d) $S(u, *[g(\overline{\omega})]) = -4 \cdot \varepsilon \cdot e \cdot \omega$.
Proof:

$$\begin{aligned} S(u, *[g(\overline{\omega})]) &= S(u, \varepsilon \cdot (-1)^{3\cdot(4-3)} \cdot \frac{4!}{(4-3)!3!} \cdot g(u) \wedge \omega) = \\ &= -4 \cdot \varepsilon \cdot S(u, g(u) \wedge \omega) = -4 \cdot \varepsilon \cdot e \cdot \omega \ . \end{aligned} \tag{186}$$

From the last expression, it follows that

$$\omega = -\frac{1}{4 \cdot \varepsilon \cdot e} \cdot S(u, *[g(\overline{\omega})]) = \omega_{kl} \cdot dx^k \wedge dx^l \ . \tag{187}$$

On the other side, we have $*[g(\overline{\omega})] = \sqrt{-d_g} \cdot \varepsilon_{ijkl} \cdot \overline{\omega}^{\overline{l}} \cdot dx^i \wedge dx^j \wedge dx^k$.
Proof:

$$\begin{aligned} *[g(\overline{\omega})] &= \sqrt{-d_g} \cdot \varepsilon_{ijkl} \cdot g^{\overline{l}\overline{m}} \cdot g_{m\overline{n}} \cdot \overline{\omega}^n \cdot dx^i \wedge dx^j \wedge dx^k = \\ &= \sqrt{-d_g} \cdot \varepsilon_{ijkl} \cdot g^{\overline{l}\overline{m}} \cdot g_{mn} \cdot \overline{\omega}^{\overline{n}} \cdot dx^i \wedge dx^j \wedge dx^k = \\ &= \sqrt{-d_g} \cdot \varepsilon_{ijkl} \cdot g^l_n \cdot \overline{\omega}^{\overline{n}} \cdot dx^i \wedge dx^j \wedge dx^k = \\ &= \sqrt{-d_g} \cdot \varepsilon_{ijkl} \cdot \overline{\omega}^{\overline{l}} \cdot dx^i \wedge dx^j \wedge dx^k \ . \end{aligned} \tag{188}$$

Then

$$S(u, *[g(\overline{\omega})]) = 3 \cdot \sqrt{-d_g} \cdot \varepsilon_{ijkl} \cdot u^{\overline{i}} \cdot \overline{\omega}^{\overline{l}} \cdot dx^j \wedge dx^k \ . \tag{189}$$

Proof:

$$\begin{aligned} S(u, *[g(\overline{\omega})]) &= S(u^m \cdot \partial_m, \sqrt{-d_g} \cdot \varepsilon_{ijkl} \cdot \overline{\omega}^{\overline{l}} \cdot dx^i \wedge dx^j \wedge dx^k) = \\ &= \sqrt{-d_g} \cdot \varepsilon_{ijkl} \cdot \overline{\omega}^{\overline{l}} \cdot u^m \cdot S(\partial_m, dx^i \wedge dx^j \wedge dx^k) \ , \end{aligned} \tag{190}$$

▲ For $S(\partial_m, dx^i \wedge dx^j \wedge dx^k)$ we obtain

$$S(\partial_m, dx^i \wedge dx^j \wedge dx^k) = f^i{}_m \cdot dx^j \wedge dx^k - f^j{}_m \cdot dx^i \wedge dx^k + f^k{}_m \cdot dx^i \wedge dx^j \ . \tag{191}$$

Proof:

$$S(\partial_m, dx^i \wedge dx^j \wedge dx^k) = f^i{}_m \cdot dx^j \wedge dx^k + (-1) \cdot dx^i \wedge S(\partial_m, dx^j \wedge dx^k) =$$

$$\begin{aligned} &= f^i{}_m \cdot dx^j \wedge dx^k - dx^i \wedge (f^j{}_m \cdot dx^k - dx^j \cdot S(\partial_m, dx^k)) = \\ &= f^i{}_m \cdot dx^j \wedge dx^k - f^j{}_m \cdot dx^i \wedge dx^k + f^k{}_m \cdot dx^i \wedge dx^j \ . \ \blacktriangle \end{aligned}$$

Therefore,

$$\begin{aligned} S(u, *[g(\overline{\omega})]) &= \sqrt{-d_g} \cdot \varepsilon_{ijkl} \cdot \overline{\omega}^{\overline{l}} \cdot u^m \cdot \\ &\cdot (f^i{}_m \cdot dx^j \wedge dx^k - f^j{}_m \cdot dx^i \wedge dx^k + f^k{}_m \cdot dx^i \wedge dx^j) \\ &= \sqrt{-d_g} \cdot (\varepsilon_{ijkl} \cdot \overline{\omega}^{\overline{l}} \cdot u^m \cdot f^i{}_m \cdot dx^j \wedge dx^k - \\ &- \varepsilon_{ijkl} \cdot \overline{\omega}^{\overline{l}} \cdot u^m \cdot f^j{}_m \cdot dx^i \wedge dx^k + \\ &+ \varepsilon_{ijkl} \cdot \overline{\omega}^{\overline{l}} \cdot u^m \cdot f^k{}_m \cdot dx^i \wedge dx^j) \ , \end{aligned}$$

$$\begin{aligned} S(u, *[g(\overline{\omega})]) &= \sqrt{-d_g} \cdot u^m \cdot \overline{\omega}^{\overline{l}} \cdot f^i{}_m \cdot (\varepsilon_{ijkl} - \varepsilon_{jikl} + \varepsilon_{jkil}) \cdot dx^j \wedge dx^k = \\ &= 3 \cdot \sqrt{-d_g} \cdot \varepsilon_{ijkl} \cdot u^{\overline{i}} \cdot \overline{\omega}^{\overline{l}} \cdot dx^j \wedge dx^k. \end{aligned} \tag{192}$$

For the rotation (vortex) velocity ω, we obtain

$$\begin{aligned} \omega &= -\frac{1}{4 \cdot \varepsilon \cdot e} \cdot S(u, *[g(\overline{\omega})]) = \omega_{jk} \cdot dx^j \wedge dx^k = \\ &= -\frac{3}{4 \cdot \varepsilon \cdot e} \cdot \sqrt{-d_g} \cdot \varepsilon_{ijkl} \cdot u^{\overline{i}} \cdot \overline{\omega}^{\overline{l}} \cdot dx^j \wedge dx^k \quad , \end{aligned} \tag{193}$$

$$\begin{aligned}\omega_{jk} &= -\frac{3}{4\cdot\varepsilon\cdot e}\cdot\sqrt{-d_g}\cdot\varepsilon_{ijkl}\cdot u^{\overline{i}}\cdot\overline{\omega}^{\overline{l}} = \\ &= -\frac{3}{4\cdot\varepsilon\cdot e}\cdot\sqrt{-d_g}\cdot\varepsilon_{iljk}\cdot u^{\overline{i}}\cdot\overline{\omega}^{\overline{l}} \quad . \end{aligned} \tag{194}$$

3. The rotation velocity tensor ω is orthogonal to u and $\overline{\omega}$. The first property $\omega(u) = -(u)(\omega) = 0$ follows from the construction of ω. The second property $\omega(\overline{\omega}) = -(\overline{\omega})(\omega) = 0$ can be proved.

Proof:

$$\begin{aligned}\omega(\overline{\omega}) &= \frac{1}{2}\cdot\omega_{jk}\cdot(dx^j\otimes dx^k - dx^k\otimes dx^j)(\overline{\omega}^l\cdot\partial_l) = \\ &= \frac{1}{2}\cdot\omega_{jk}\cdot\overline{\omega}^l\cdot(f^k{}_l\cdot dx^j - f^j{}_l\cdot dx^k) = \\ &= \frac{1}{2}\cdot(\omega_{jk}\cdot\overline{\omega}^l\cdot f^k{}_l\cdot dx^j - \omega_{jk}\cdot\overline{\omega}^l\cdot f^j{}_l\cdot dx^k) = \\ &= \frac{1}{2}\cdot(\omega_{jk}\cdot\overline{\omega}^{\overline{k}}\cdot dx^j - \omega_{jk}\cdot\overline{\omega}^{\overline{j}}\cdot dx^k) = \\ &= \frac{1}{2}\cdot(\omega_{jk}\cdot\overline{\omega}^{\overline{k}}\cdot dx^j - \omega_{kj}\cdot\overline{\omega}^{\overline{k}}\cdot dx^j) = \\ &= \frac{1}{2}\cdot(\omega_{jk}+\omega_{jk})\cdot\overline{\omega}^{\overline{k}}\cdot dx^j = \\ &= \omega_{jk}\cdot\overline{\omega}^{\overline{k}}\cdot dx^j = \\ &= -\frac{3}{4\cdot\varepsilon\cdot e}\cdot\sqrt{-d_g}\cdot\varepsilon_{iljk}\cdot u^{\overline{i}}\cdot\overline{\omega}^{\overline{l}}\cdot\overline{\omega}^{\overline{k}}\cdot dx^j = 0 \ . \end{aligned} \tag{195}$$

Therefore, the rotation (vortex) velocity vector $\overline{\omega}$ is orthogonal to the velocity vector u and to the rotation velocity tensor ω.

Special case: $(\overline{L}_n, g)$-space admitting the conditions $\sigma = 0$, $\theta = 0$.

Since $g(u,\overline{\omega}) = 0$, we can chose u and $\overline{\omega}$ as tangent vectors to the two of the co-ordinate lines, i.e. u and $\overline{\omega}$ could fulfil the condition $\pounds_u\overline{\omega} = 0$. Then

$$h_u(\nabla_u\overline{\omega}) = \omega(\overline{\omega}) = 0\ , \qquad {}_{rel}v = \overline{g}[h_u(\nabla_u\overline{\omega})] = 0 \quad . \tag{196}$$

On the other side,

$$\nabla_u\overline{\omega} = \frac{1}{e}\cdot g(u,\nabla_u\overline{\omega})\cdot u + {}_{rel}v = \frac{1}{e}\cdot g(u,\nabla_u\overline{\omega})\cdot u\ , \tag{197}$$

$$\begin{aligned} g(u,\nabla_u\overline{\omega}) &= \nabla_u[g(u,\overline{\omega})] - g(\nabla_u u,\overline{\omega}) - (\nabla_u g)(u,\overline{\omega})\ , \quad g(u,\overline{\omega}) = 0\ , \\ \nabla_u[g(u,\overline{\omega})] &= u[g(u,\overline{\omega})] = 0\ , \end{aligned} \tag{198}$$

$$\begin{aligned} \nabla_u\overline{\omega} &= -\frac{1}{e}\cdot[g(\nabla_u u,\overline{\omega}) + (\nabla_u g)(u,\overline{\omega})]\cdot u\ , \quad \nabla_u u = a\ , \\ \nabla_u\overline{\omega} &= -\frac{1}{e}\cdot[g(a,\overline{\omega}) + (\nabla_u g)(u,\overline{\omega})]\cdot u\ . \end{aligned} \tag{199}$$

The rotation vector $\overline{\omega}$ does not change in directions, orthogonal to u. Its change along u is collinear to u and depends on $\nabla_u g$ and on the acceleration a.

Special case: Weyl's spaces. In Weyl's spaces (with or without torsion), where $\nabla_\xi g = (1/n)\cdot Q_\xi\cdot g$ for $\forall\xi\in T(M)$ and $(\nabla_u g)(u,\overline{\omega}) = (1/n)\cdot Q_\xi\cdot g(u,\overline{\omega}) = 0$, the rotation vector $\overline{\omega}$ does not change along u if u is a tangent vector on an auto-parallel curve, i.e. if $\nabla_u u = a = 0$,

$$\nabla_u\overline{\omega} = 0\ , \qquad a := 0\ , \tag{200}$$

i.e. $\overline{\omega}$ is transported parallel along u.

4. Change of the velocity vector u along the rotation (vortex) velocity vector $\overline{\omega}$.

The change of the velocity u along $\overline{\omega}$ could be represented in the form

$$\begin{aligned}\nabla_{\overline{\omega}}u &= \overline{g}({}_sE)(\overline{\omega}) + \overline{g}(S)(\overline{\omega}) + \frac{1}{n-1}\cdot\overline{g}[E]\cdot\overline{g}[h_u(\overline{\omega})] + \\ &\quad + \frac{1}{2\cdot e}\cdot[\overline{\omega}e - (\nabla_{\overline{\omega}}g)(u,u)]\cdot u\ . \end{aligned} \tag{201}$$

Since $\overline{g}[h_u(\overline{\omega})] = \overline{\omega}$ and $\overline{g}[E] = \theta_o$, we can also write

$$\begin{aligned} \nabla_{\overline{\omega}} u &= \overline{g}({}_sE)(\overline{\omega}) + \overline{g}(S)(\overline{\omega}) + \frac{1}{n-1} \cdot \theta_o \cdot \overline{\omega} + \\ &+ \frac{1}{2 \cdot e} \cdot [\overline{\omega} e - (\nabla_{\overline{\omega}} g)(u,u)] \cdot u \ . \end{aligned} \tag{202}$$

Special case: Shear-free ($\sigma = {}_sE = 0$) and expansion-free ($\overline{g}[E] = \theta = 0$) $\overline{V}_n$-spaces with $\omega = S$ and $\nabla_\xi g = 0$ for $\forall \xi \in T(M)$, $e :=$const.$\neq 0$. For these types of spaces $\nabla_{\overline{\omega}} u = 0$. Therefore, in $\overline{U}_n$- and $\overline{V}_n$-spaces the velocity u does not change along the rotation (vortex) velocity vector $\overline{\omega}$. This means that all particles (material points, material elements) lying on an axis, collinear to $\overline{\omega}$, have one and the same velocity which remains unchanged along this axis

$$\nabla_{\overline{\omega}}[g(u,u)] = \overline{\omega} e = (\nabla_{\overline{\omega}} g)(u,u) + 2 \cdot g(\nabla_{\overline{\omega}} u, u) = 0 \quad . \tag{203}$$

and u is parallel transported along u. This fact is related to the physical interpretation of $\overline{\omega}$ as a rotation axis.

In $(\overline{L}_n, g)$- and (L_n, g)-spaces, the rotation velocity vector $\overline{\omega}$ changes in general along the velocity u. The same is valid for the velocity u along the rotation (vortex) vector $\overline{\omega}$. This means that every material point in the flow could have its own rotation (vortex) velocity vector $\overline{\omega}$ different from that of the other material points in its neighborhoods and even different from the rotation vector of the points lying on the $\overline{\omega}$ itself. In general, we have the relation

$$\pounds_u \overline{\omega} = \nabla_u \overline{\omega} - \nabla_{\overline{\omega}} u - T(u, \overline{\omega}) \ . \tag{204}$$

On the other side, we can calculate the change of the velocity vector u by its transport along the rotation (vortex) velocity vector $\overline{\omega}$. From the relations

$$\nabla_u \overline{\omega} = \frac{1}{e} \cdot g(u, \nabla_u \overline{\omega}) \cdot u + {}_{rel}v \ , \tag{205}$$

$$g(u, \nabla_u \overline{\omega}) = -[g(a, \overline{\omega}) + (\nabla_u g)(\overline{\omega}, u)] \ , \tag{206}$$

$$\begin{aligned} {}_{rel}v &= \overline{g}[h_u(\nabla_u \overline{\omega})] = \overline{g}(h_u)(\frac{l}{e} \cdot a - \pounds_{\overline{\omega}} u) + \overline{g}[d(\overline{\omega})] = \\ &= \overline{g}(h_u)(\pounds_u \overline{\omega}) + \overline{g}[d(\overline{\omega})] \quad , \qquad l = g(u, \overline{\omega}) = 0 \ , \end{aligned} \tag{207}$$

$$\begin{aligned} \nabla_u \overline{\omega} &= -\frac{1}{e} \cdot [g(a, \overline{\omega}) + (\nabla_u g)(\overline{\omega}, u)] \cdot u + \overline{g}(h_u)(\pounds_u \overline{\omega}) + \overline{g}[d(\overline{\omega})] = \\ &= -\frac{1}{e} \cdot [g(a, \overline{\omega}) + (\nabla_u g)(\overline{\omega}, u)] \cdot u + \overline{g}(h_u)(\pounds_u \overline{\omega}) + \\ &+ \overline{g}[\sigma + \omega + \frac{1}{n-1} \cdot \theta \cdot h_u](\overline{\omega}) \quad , \end{aligned} \tag{208}$$

$$h_u(\overline{\omega}) = g(\overline{\omega}) \ , \qquad \omega(\overline{\omega}) = 0 \ , \qquad h_u(\overline{g}) h_u = (h_u) \overline{g}(h_u) = h_u \ , \tag{209}$$

it follows for $\nabla_u \overline{\omega}$

$$\begin{aligned} \nabla_u \overline{\omega} &= -\frac{1}{e} \cdot [g(a, \overline{\omega}) + (\nabla_u g)(\overline{\omega}, u)] \cdot u + (\pounds_u \overline{\omega})_\perp + \\ &+ \overline{g}(\sigma)(\overline{\omega}) + \frac{1}{n-1} \cdot \theta \cdot \overline{\omega} \quad , \\ (\pounds_u \overline{\omega})_\perp \quad &: \quad = \overline{g}(h_u)(\pounds_u \overline{\omega}) \end{aligned} \tag{210}$$

If we substitute now the expressions for $\nabla_u \overline{\omega}$ and for $\nabla_{\overline{\omega}} u$ in $\pounds_u \overline{\omega} = \nabla_u \overline{\omega} - \nabla_{\overline{\omega}} u - T(u, \overline{\omega})$ then we will obtain the explicit form of the torsion vector $T(u, \overline{\omega})$

$$\begin{aligned} T(u, \overline{\omega}) &= -\frac{1}{e} \cdot \{g(a, \overline{\omega}) + (\nabla_u g)(\overline{\omega}, u) + g(u, \pounds_u \overline{\omega}) + \\ &+ \frac{1}{2} \cdot [\overline{\omega} e - (\nabla_{\overline{\omega}} g)(u,u)]\} \cdot u - \\ &- [\overline{g}(\sigma_1)(\overline{\omega}) + \overline{g}(\omega_1)(\overline{\omega}) + \frac{1}{n-1} \cdot \theta_1 \cdot \overline{\omega}] \ , \end{aligned} \tag{211}$$

where

$$\omega = \omega_o - \omega_1 \quad , \quad \omega_o = S \; , \quad \omega_1 = Q \; , \qquad \omega(\overline{\omega}) = 0 = \omega_o(\overline{\omega}) - \omega_1(\overline{\omega}) \; ,$$
$$\sigma = \sigma_o - \sigma_1 \quad , \sigma_o = {}_sE \; , \quad \sigma_1 = {}_sP \; .$$

Special case: If u and $\overline{\omega}$ are tangent vectors to co-ordinate lines in M, then $\pounds_u\overline{\omega} = 0$ and we can find a representation of the torsion vector $T(u, \overline{\omega})$ in the form

$$T(u, \overline{\omega}) = \nabla_u \overline{\omega} - \nabla_{\overline{\omega}} u \; . \tag{212}$$

Special case: $\overline{U}_n$-spaces: $\nabla_\xi g = 0$ for $\forall \xi \in T(M)$, $n = 4$, $\pounds_u\overline{\omega} = 0$.

$$T(u, \overline{\omega}) = -\frac{1}{e} \cdot [g(a, \overline{\omega}) + \frac{1}{2} \cdot \overline{\omega}e] \cdot u - \overline{g}[d_1(\overline{\omega})] \quad . \tag{213}$$

If $T(u, \overline{\omega}) := 0$, then

$$\overline{g}[d_1(\overline{\omega})] = -\frac{1}{e} \cdot [g(a, \overline{\omega}) + \frac{1}{2} \cdot \overline{\omega}e] \cdot u \quad . \tag{214}$$

Since $d_1(u) = 0$ and $[g(u)](\overline{g})[d_1(\overline{\omega})] = d_1(u, \overline{\omega}) = 0$, we obtain

$$g(a, \overline{\omega}) + \frac{1}{2} \cdot \overline{\omega}e = 0 \quad , \quad \overline{\omega}e = -2 \cdot g(a, \overline{\omega}) \quad . \tag{215}$$

In this special case [U_n-space, $n = 4$, $\pounds_u\overline{\omega} = 0$, $T(u, \overline{\omega}) = 0$] the velocity u will change along the axis $\overline{\omega}$ only if $g(a, \overline{\omega}) = 0$. This means that the condition $\overline{\omega}e = 0$ will be fulfilled only if the acceleration a is orthogonal to $\overline{\omega}$ [$g(a, \overline{\omega}) = 0$] or $a = 0$ (if $\overline{\omega} \neq 0$). The last condition is fulfilled if the material points are moving on auto-parallel trajectories and the same time having vortex velocity $\overline{\omega} \neq 0$.

Special case: V_n-spaces. $n = 4$, $\pounds_u\overline{\omega} = 0$, $T(\xi, \eta) := 0$ for $\forall \xi$, $\eta \in T(M)$.

$$\overline{\omega}e = -2 \cdot \; g(a, \overline{\omega}) \; . \tag{216}$$

If further $e := \text{const.} \neq 0 : g(a, \overline{\omega}) = 0$.

In the Einstein theory of gravitation (where $e := \text{const.} \neq 0$) the vortex velocity vector $\overline{\omega}$ is always orthogonal to the acceleration a or the acceleration a is equal to zero (auto-parallel, geodesic trajectories). Since in the general case $g(a, u) = 0$, and $g(u, \overline{\omega}) = 0$, the vectors u, $\overline{\omega}$, and a construct a triad (3-Bein), where u is a time-like vector, where $\overline{\omega}$ and a are space-like vectors.

We can introduce abbreviations for the following invariants:

$$\begin{aligned}
g(\overline{\omega}, \overline{\omega}) \quad &: \quad = \overline{\omega}^2 = \pm \, l^2_{\overline{\omega}} \qquad , \qquad l_{\overline{\omega}} = \mid g(\overline{\omega}, \overline{\omega}) \mid^{1/2} \quad , \\
\overline{g}[\omega(\overline{g})\omega] \quad &= \quad \omega_{ij} \cdot g^{\overline{jk}} \cdot \omega_{kl} \cdot g^{\overline{il}} := \omega^2 \quad , \\
\overline{g}[\sigma(\overline{g})\sigma] \quad &= \quad \sigma_{ik} \cdot g^{\overline{kl}} \cdot \sigma_{lj} \cdot g^{\overline{ij}} := \sigma^2 \quad , \\
\overline{g}[\sigma(\overline{g})\sigma(\overline{g})\sigma] \quad &= \quad \sigma_{ik} \cdot g^{\overline{kl}} \cdot \sigma_{lm} \cdot g^{\overline{mn}} \cdot \sigma_{nj} \cdot g^{\overline{ij}} := \sigma^3 \quad , \\
g(a, a) \quad &: \quad = a^2 = \pm l^2_a \quad , \quad l_a = \mid g(a, a) \mid^{1/2} \quad .
\end{aligned} \tag{217}$$

Let us now consider the change of the vector u along the curve $x^i(\tau_0 = \text{const.}, \lambda^a)$.

3 Friction velocity. Deformation friction velocity, shear friction velocity, rotation (vortex) friction velocity, and expansion friction velocity

3.1 Friction velocity

In the case of change of the vector u along the curve $x^i(\tau_0 = \text{const.}, \lambda^a)$, we have

$$\left(\frac{Du}{d\lambda^a}\right)_{(\tau_0, \lambda^a_0)} = \left(\nabla_{\xi_{(a)\perp}} u\right)_{(\tau_0, \lambda^a_0)} = \lim_{d\lambda^a \to 0} \frac{u_{(\tau_0, \lambda^a_0 + d\lambda^a)} - u_{(\tau_0, \lambda^a_0)}}{d\lambda^a} \quad , \quad \xi_{(a)\perp} = \frac{d}{d\lambda^a} \quad . \tag{218}$$

The vector $\nabla_{\xi_{(a)\perp}} u$ describes the change of the velocity u of material points along the orthogonal to u curves $x^i(\tau_0, \lambda^a)$. This means that $\nabla_{\xi_{(a)\perp}} u$ shows how the velocity of the material elements changes at a cross-section of a flow. Usually, the change of the velocity of material points in a direction, orthogonal to the velocity, is related to the existence of inner friction (viscosity of the media) between the different current lines of the flow.

In an analogous way as for $\nabla_u \xi_{(a)\perp}$ the vector $\nabla_{\xi_{(a)\perp}} u$ can be decomposed in two parts: one collinear to $\xi_{(a)\perp}$ and one orthogonal to $\xi_{(a)\perp}$, i.e.

$$\begin{aligned}
\frac{Du}{d\lambda^a} &= \nabla_{\xi_{(a)\perp}} u = \frac{g(\nabla_{\xi_{(a)\perp}} u, \xi_{(a)\perp})}{g(\xi_{(a)\perp}, \xi_{(a)\perp})} \cdot \xi_{(a)\perp} + R_{u(a)} = \\
&= \frac{\widetilde{l}_{(a)}}{\pm l^2_{\xi(a)\perp}} \cdot \xi_{(a)\perp} + R_{u(a)} \quad , \qquad (219) \\
\widetilde{l}_{(a)} &= g(\nabla_{\xi_{(a)\perp}} u, \xi_{(a)\perp}) \quad , \quad \pm l^2_{\xi(a)\perp} = g(\xi_{(a)\perp}, \xi_{(a)\perp}) \quad , \\
R_{u(a)} &= \overline{g}[h_{\xi_{(a)\perp}}(\nabla_{\xi_{(a)\perp}} u)] = \overline{g}[h_{\xi_{(a)\perp}}(\frac{Du}{d\lambda^a})] \quad , \\
g(R_{u(a)}, \xi_{(a)\perp}) &= 0 \quad , \quad h_{\xi_{(a)\perp}} = g - g(\xi_{(a)\perp}) \otimes g(\xi_{(a)\perp}) \quad .
\end{aligned}$$

The vector $R_{u(a)}$ is called *friction velocity vector* or *friction velocity*. It could be consider as a measure for the friction between the layers of a flow. We will consider later the structure of the friction vector $R_{u(a)}$.

3.2 Deformation friction velocity, shear friction velocity, rotation (vortex) friction velocity, and expansion friction velocity

Let us now consider a vector field $\xi_\perp$ with $g(u, \xi_\perp) = 0$, $\pounds_{\xi_\perp} u = -\pounds_u \xi_\perp = 0$. Then we have the relations

$$\begin{aligned}
\nabla_{\xi_\perp} u &= \nabla_u \xi_\perp - \pounds_u \xi_\perp - T(u, \xi_\perp) = ({}_\xi k) g(u) - \pounds_u \xi_\perp \quad , \\
({}_\xi k) g(u) &= {}_\xi k[g(u)] = \nabla_u \xi_\perp - T(u, \xi_\perp) = (\xi^i_{\perp ;j} \cdot u^j - T_{kl}{}^i \cdot u^k \cdot \xi^l_\perp) \cdot \partial_i = \\
&= (\xi^i_{\perp ;l} - T_{lk}{}^i \cdot \xi^k_\perp) \cdot u^l \cdot \partial_i = (\xi^i_{\perp ;l} - T_{lk}{}^i \cdot \xi^k_\perp) \cdot g^{lj} \cdot g_{\overline{j}\overline{m}} \cdot u^m \cdot \partial_i = \\
&= {}_\xi k^{ij} \cdot g_{\overline{j}\overline{m}} \cdot u^m \cdot \partial_i = {}_\xi k(g)(u) = ({}_\xi k) g(u) = {}_\xi k[g(u)] \quad , \qquad (220) \\
{}_\xi k &= {}_\xi k^{ij} \cdot \partial_i \otimes \partial_j \quad , \qquad {}_\xi k^{ij} = (\xi^i_{\perp ;l} - T_{lk}{}^i \cdot \xi^k_\perp) \cdot g^{lj} \cdot \partial_i \otimes \partial_j \quad , \\
{}_\xi k(g)(u) &= {}_\xi k(h_{\xi_\perp} + \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot g(\xi_\perp) \otimes g(\xi_\perp))(u) = ({}_\xi k) h_{\xi_\perp}(u) \quad , \\
\nabla_{\xi_\perp} u &= -\pounds_u \xi_\perp + ({}_\xi k) h_{\xi_\perp}(u) \quad .
\end{aligned}$$

On the other side, from the relations

$$\nabla_{\xi_\perp} u = \frac{g(\nabla_{\xi_\perp} u, u)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + R_u \quad , \qquad R_u = \overline{g}[h_{\xi_\perp}(\nabla_{\xi_\perp} u)] \quad , \qquad (221)$$

$$h_{\xi_\perp}(\xi_\perp) = 0 \quad , \qquad h^{\xi_\perp} = \overline{g} - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp \otimes \xi_\perp \quad , \qquad (222)$$

$$g(R_u) = h_{\xi_\perp}(\nabla_{\xi_\perp} u) \quad , \qquad (223)$$

it follows the expression

$$\begin{aligned}
h_{\xi_\perp}(\nabla_{\xi_\perp} u) &= -h_{\xi_\perp}(\pounds_u \xi_\perp) + h_{\xi_\perp}({}_\xi k) h_{\xi_\perp}(u) = \\
&= -h_{\xi_\perp}(\pounds_u \xi_\perp) + R(u) \quad , \qquad (224) \\
R(u) &= h_{\xi_\perp}({}_\xi k) h_{\xi_\perp}(u) \quad , \qquad (225)
\end{aligned}$$

[compare with $h_u(\nabla_u \xi) = h_u(\frac{l}{e} \cdot a - \pounds_\xi u) + h_u(k) h_u(\xi)$ for $l = 0$].

The tensor of second rank R is called *friction deformation velocity tensor*. It can be represented in the form analogous of the form of the deformation velocity tensor

$$R = {}_\sigma R + {}_\omega R + \frac{1}{n-1} \cdot {}_\theta R \cdot h_{\xi_\perp} \quad . \qquad (226)$$

The tensors ${}_{\sigma}R$, ${}_{\omega}R$, and the invariant ${}_{\theta}R$ could be found in analogous way as the tensors σ, ω, and the invariant θ.

The symmetric trace-free tensor ${}_{\sigma}R$ has the form

$$
\begin{aligned}
{}_{\sigma}R &= {}_{\xi s}E - {}_{\xi s}P = {}_{\xi}E - {}_{\xi}P - \frac{1}{n-1} \cdot \overline{g}[{}_{\xi}E - {}_{\xi}P] \cdot h_{\xi_{\perp}} = {}_{\sigma}R_{ij} \cdot dx^{i}.dx^{j} = \\
&= {}_{\sigma}R_{ij} \cdot e^{i}.e^{j} \quad , \\
{}_{\xi s}E &= {}_{\xi}E - \frac{1}{n-1} \cdot \overline{g}[{}_{\xi}E] \cdot h_{\xi_{\perp}} \quad , \qquad \overline{g}[{}_{\xi}E] = g^{ij} \cdot {}_{\xi}E_{\overline{ij}} = g^{\overline{ij}} \cdot {}_{\xi}E_{ij} = {}_{\theta_o}R \quad , \\
{}_{\theta_o}R &= \xi^{n}_{\perp;n} - \frac{1}{2 \cdot e_{\xi_{\perp}}} \cdot (e_{\xi_{\perp},k} \cdot \xi^{k}_{\perp} - g_{kl;m} \cdot \xi^{m}_{\perp} \cdot \xi^{\overline{k}}_{\perp} \cdot \xi^{\overline{l}}_{\perp}) \quad , \qquad (227) \\
e_{\xi_{\perp}} &= g(\xi_{\perp}, \xi_{\perp}) = \pm l^{2}_{\xi_{\perp}} \quad , \\
{}_{\xi}E &= h_{\xi_{\perp}}({}_{\xi}\varepsilon)h_{\xi_{\perp}} \quad , \qquad {}_{\xi}k_{s} = {}_{\xi}\varepsilon - {}_{\xi}m \quad , \qquad e_{\xi_{\perp},k} = \partial_{k}e_{\xi_{\perp}} = e_{k}(e_{\xi_{\perp}}) \quad , \\
{}_{\xi}\varepsilon &= \frac{1}{2} \cdot (\xi^{i}_{\perp;l} \cdot g^{lj} + \xi^{j}_{\perp;l} \cdot g^{li}) \cdot \partial_{i}.\partial j \quad , \\
{}_{\xi}m &= \frac{1}{2} \cdot (T_{lk}{}^{i} \cdot \xi^{k}_{\perp} \cdot g^{lj} + T_{lk}{}^{j} \cdot \xi^{k}_{\perp} \cdot g^{li}) \cdot \partial_{i}.\partial j \quad .
\end{aligned}
$$

The symmetric trace-free tensor ${}_{\xi s}E$ is the torsion-free shear friction velocity tensor (shear friction), the symmetric trace-free tensor ${}_{\xi s}P$ is the shear friction velocity tensor induced by the torsion,

$$
\begin{aligned}
{}_{\xi s}P &= {}_{\xi}P - \frac{1}{n-1} \cdot \overline{g}[{}_{\xi}P] \cdot h_{\xi_{\perp}} \quad , \qquad \overline{g}[{}_{\xi}P] = g^{ij} \cdot {}_{\xi}P_{\overline{ij}} = g^{\overline{ij}} \cdot {}_{\xi}P_{ij} = {}_{\theta_1}R \quad , \qquad (228) \\
{}_{\xi}P &= h_{\xi_{\perp}}({}_{\xi}m)h_{\xi_{\perp}} \quad , \qquad {}_{\theta_1}R = T_{kl}{}^{k} \cdot \xi^{l}_{\perp} \quad , \qquad (229) \\
{}_{\theta}R &= {}_{\theta_0}R - {}_{\theta_1}R \quad . \qquad (230)
\end{aligned}
$$

The invariant ${}_{\theta}R$ is the expansion friction velocity (expansion friction), the invariant${}_{\theta_o}R$ is the torsion-free expansion friction velocity, the invariant ${}_{\theta_1}R$ is the expansion friction velocity induced by the torsion.

The antisymmetric tensor of second rank ${}_{\omega}R$ is the rotation (vortex) friction velocity tensor (rotation friction)

$$
\begin{aligned}
{}_{\omega}R &= h_{\xi_{\perp}}({}_{\xi}k_{a})h_{\xi_{\perp}} = h_{\xi_{\perp}}({}_{\xi}s)h_{\xi_{\perp}} - h_{\xi_{\perp}}({}_{\xi}q)h_{\xi_{\perp}} = {}_{\xi}S - {}_{\xi}Q \quad , \qquad (231) \\
{}_{\xi}S &= h_{\xi_{\perp}}({}_{\xi}s)h_{\xi_{\perp}} \quad , \qquad {}_{\xi}Q = h_{\xi_{\perp}}({}_{\xi}q)h_{\xi_{\perp}} \quad , \qquad (232) \\
{}_{\xi}s &= \frac{1}{2} \cdot (\xi^{k}_{\perp;m} \cdot g^{ml} - \xi^{l}_{\perp;m} \cdot g^{mk}) \cdot \partial_{i} \wedge \partial j \quad , \qquad (233) \\
{}_{\xi}q &= \frac{1}{2} \cdot (T_{mn}{}^{k} \cdot \xi^{n}_{\perp} \cdot g^{ml} - T_{mn}{}^{l} \cdot \xi^{n}_{\perp} \cdot g^{mk}) \cdot \partial_{i} \wedge \partial j \quad .
\end{aligned}
$$

The antisymmetric tensor ${}_{\xi}S$ is the torsion-free rotation (vortex) friction velocity tensor, the antisymmetric tensor ${}_{\xi}Q$ is the rotation (vortex) friction velocity tensor induced by the torsion.

By means of the expressions for ${}_{\sigma}R$, ${}_{\omega}R$, and ${}_{\theta}R$ the friction deformation velocity tensor R could be written in the form

$$R = {}_{0}R - {}_{T}R \ , \qquad (234)$$

where

$$
\begin{aligned}
{}_{0}R &= {}_{\xi s}E + {}_{\xi}S + \frac{1}{n-1} \cdot {}_{\theta_o}R \cdot h_{\xi_{\perp}} \ , \qquad (235) \\
{}_{T}R &= {}_{\xi s}P + {}_{\xi}Q + \frac{1}{n-1} \cdot {}_{\theta_1}R \cdot h_{\xi_{\perp}} \quad . \qquad (236)
\end{aligned}
$$

The tensor ${}_{0}R$ is the torsion-free friction deformation velocity tensor and the tensor ${}_{T}R$ is the friction deformation velocity induced by the torsion. for the case of V_n-spaces, ${}_{T}R = 0$ (${}_{\xi s}P = 0$, ${}_{\xi}Q = 0$, ${}_{\theta_1}R = 0$).

In an analogous way as in the case of the shear velocity tensor σ and the expansion velocity invariant θ, the shear friction velocity tensor ${}_{\sigma}R$ and the expansion friction invariant ${}_{\theta}R$ could be represented in the forms

$$
\begin{aligned}
{}_{\sigma}R &= \frac{1}{2} \cdot \{h_{\xi_{\perp}}(\nabla_{\xi_{\perp}}\overline{g} - \pounds_{\xi_{\perp}}\overline{g})h_{\xi_{\perp}} - \frac{1}{n-1} \cdot (h_{\xi_{\perp}}[\nabla_{\xi_{\perp}}\overline{g} - \pounds_{\xi_{\perp}}\overline{g}])h_{\xi_{\perp}} \quad , \qquad (237) \\
{}_{\theta}R &= \frac{1}{2} \cdot h_{\xi_{\perp}}[\nabla_{\xi_{\perp}}\overline{g} - \pounds_{\xi_{\perp}}\overline{g}] \quad . \qquad (238)
\end{aligned}
$$

3.3 Representation of the friction velocity by the use of the kinematic characteristics of the relative velocity

The relative velocity tensor and the friction velocity tensor can be related to each other on the basis of the relation $\pounds_u \xi = \nabla_u \xi - \nabla_\xi u - T(u, \xi)$. Let us now consider the representation of $\nabla_\xi u$ by the use of the corresponding to u projective metrics h_u and h^u. We can write for $\nabla_\xi u$

$$\begin{aligned}
\nabla_\xi u &= u^i{}_{;j} \cdot u^j \cdot \partial_i = u^\alpha{}_{/\beta} \cdot \xi^\beta \cdot e_\alpha = \\
&= (\varepsilon + s) g(\xi) = (\varepsilon + s)[g(\xi)] = \\
&= (\varepsilon + s)[h_u + \frac{1}{e} \cdot g(u) \otimes g(u)](\xi) = \\
&= (\varepsilon + s) h_u(\xi) + \frac{1}{e} \cdot (\varepsilon + s)[g(u)] \otimes [g(u)](\xi) = \\
&= \{(\varepsilon + s) h_u + \frac{1}{e} \cdot (\varepsilon + s)[g(u)] \otimes [g(u)]\}(\xi) \quad ,
\end{aligned} \tag{239}$$

$$\begin{aligned}
g(\nabla_\xi u) &= \{g(\varepsilon + s) h_u + \frac{1}{e} \cdot g(\varepsilon + s)[g(u)] \otimes [g(u)]\}(\xi) = \\
&= \{[h_u + \frac{1}{e} \cdot g(u) \otimes g(u)](\varepsilon + s) h_u + \frac{1}{e} \cdot g(\varepsilon + s)[g(u)] \otimes [g(u)]\}(\xi) = \\
&= \{h_u(\varepsilon + s) h_u + \frac{1}{e} \cdot g(u) \otimes [g(u)](\varepsilon + s) h_u + \\
&\quad + \frac{1}{e} \cdot g(\varepsilon + s)[g(u)] \otimes [g(u)]\}(\xi) \quad .
\end{aligned} \tag{240}$$

Since

$$\begin{aligned}
h_u(\varepsilon + s) h_u &= h_u(\varepsilon) h_u + h_u(s) h_u = E + S = \\
&= {}_sE + S + \frac{1}{n-1} \cdot \overline{g}[E] \cdot h_u = \\
&= {}_sE + S + \frac{1}{n-1} \cdot \theta_o \cdot h_u = d_o \quad , \\
(\varepsilon + s)[g(u)] &= a = \nabla_u u \quad , \\
[g(u)](\varepsilon + s) h_u &= [g(u)](\varepsilon + s) g - \frac{1}{e} \cdot g(u, a) \cdot g(u) \quad , \\
[g(u)](\varepsilon + s) g(\xi) &= \frac{1}{2} \cdot [\xi e - (\nabla_\xi g)(u, u)] \quad ,
\end{aligned}$$

we obtain for $g(\nabla_\xi u)$ and $\nabla_\xi u$ respectively

$$\begin{aligned}
g(\nabla_\xi u) &= \{d_o + \frac{1}{e} \cdot g(a) \otimes g(u) - \\
&\quad - \frac{1}{2 \cdot e^2} \cdot [ue - (\nabla_u g)(u, u)] \cdot g(u) \otimes g(u)\}(\xi) + \\
&\quad + \frac{1}{2 \cdot e} \cdot [\xi e - (\nabla_\xi g)(u, u)] \cdot g(u) \quad , \\
g(\nabla_\xi u) &= \{\frac{1}{e} \cdot g(a) \otimes g(u) + {}_sE + S + \frac{1}{n-1} \cdot \overline{g}[E] \cdot h_u\}(\xi) + \\
&\quad + \frac{1}{2 \cdot e} \cdot [\xi e - (\nabla_\xi g)(u, u)] \cdot g(u) - \\
&\quad - \frac{1}{2 \cdot e^2} \cdot \{[ue - (\nabla_u g)(u, u)] \cdot g(u) \otimes g(u)\}(\xi)
\end{aligned} \tag{241}$$

$$\begin{aligned}
\nabla_\xi u &= \{\overline{g}(d_o) + \frac{1}{e} \cdot a \otimes g(u) - \frac{1}{2 \cdot e^2} \cdot [ue - (\nabla_u g)(u, u)] \cdot u \otimes g(u)\}(\xi) + \\
&\quad + \frac{1}{2 \cdot e} \cdot [\xi e - (\nabla_\xi g)(u, u)] \cdot u \quad , \\
\nabla_\xi u &= \{\frac{1}{e} \cdot a \otimes g(u) + \overline{g}({}_sE) + \overline{g}(S) + \frac{1}{n-1} \cdot \overline{g}[E] \cdot \overline{g}(h_u)\}(\xi) +
\end{aligned} \tag{242}$$

$$+\frac{1}{2\cdot e}\cdot[\xi e-(\nabla_\xi g)(u,u)]\cdot u-$$
$$-\frac{1}{2\cdot e^2}\cdot\{[ue-(\nabla_u g)(u,u)]\cdot u\otimes g(u)\}(\xi)\quad . \tag{243}$$

The relations for $g(\nabla_\xi u)$ and $\nabla_\xi u$ are valid for an arbitrary given contravariant vector field $\xi\in T(M)$. For an orthogonal to u vector field $\xi_\perp, [g(u,\xi_\perp)=0]$, we obtain the representations for $g(\nabla_\xi u)$ and $\nabla_\xi u$ in the forms

$$g(\nabla_{\xi_\perp}u) = d_o(\xi_\perp)+\frac{1}{2\cdot e}\cdot[\xi_\perp e-(\nabla_{\xi_\perp}g)(u,u)]\cdot g(u)\quad , \tag{244}$$
$$\nabla_{\xi_\perp}u = \overline{g}(d_o)(\xi_\perp)+\frac{1}{2\cdot e}\cdot[\xi_\perp e-(\nabla_{\xi_\perp}g)(u,u)]\cdot u\quad .$$

Therefore, for $\nabla_{\xi_\perp}u$ and $h_{\xi_\perp}(\nabla_{\xi_\perp}u)$ we have the relations

$$\nabla_{\xi_\perp}u = \frac{g(\nabla_{\xi_\perp}u,u)}{g(\xi_\perp,\xi_\perp)}\cdot\xi_\perp+R_u =$$
$$= \overline{g}(d_o)(\xi_\perp)+\frac{1}{2\cdot e}\cdot[\xi_\perp e-(\nabla_{\xi_\perp}g)(u,u)]\cdot u\quad , \tag{245}$$
$$h_{\xi_\perp}(\nabla_{\xi_\perp}u) = h_{\xi_\perp}(\overline{g})(d_o)(\xi_\perp)+\frac{1}{2\cdot e}\cdot[\xi_\perp e-(\nabla_{\xi_\perp}g)(u,u)]\cdot g(u)\quad , \tag{246}$$
$$h_{\xi_\perp}(u) = g(u)-\frac{1}{e_{\xi_\perp}}\cdot g(\xi_\perp,u)\cdot g(\xi_\perp)=g(u)$$
$$R_u = \overline{g}[h_{\xi_\perp}(\nabla_{\xi_\perp}u)]=(\overline{g})h_{\xi_\perp}(\overline{g})(d_o)(\xi_\perp)+\frac{1}{2\cdot e}\cdot[\xi_\perp e-(\nabla_{\xi_\perp}g)(u,u)]\cdot u=$$
$$= h^{\xi_\perp}(d_o)(\xi_\perp)+\frac{1}{2\cdot e}\cdot[\xi_\perp e-(\nabla_{\xi_\perp}g)(u,u)]\cdot u\quad , \tag{247}$$
$$h^{\xi_\perp} = (\overline{g})h_{\xi_\perp}(\overline{g})\quad .$$

In our further consideration we will assume the existence of a proper frame of reference in a flow. From this point of view it is possible to introduce particular designations for some types of flows.

4 Special types of flows

4.1 Inertial flow

Definition 1 *A flow which material points are moving on auto-parallel lines, i.e. a flow with $\nabla_u u=a=0$ as a kinematic characteristic, is called inertial flow.*

Inertial flows will be considered below with respect to their relative accelerations.

4.2 Vortex-free (irrotational) flow

Definition 2 *A flow for which the vector field u fulfills the condition $\omega=0$ is called vortex-free (irrotational) flow.*

If we consider the explicit form for ω

$$\omega=h_u(k_a)h_u \tag{248}$$

we can prove the following propositions:

Proposition 4 *The necessary and sufficient condition for the existence of a contravariant non-null vector field with vanishing rotation velocity ($\omega=0$) is the condition*

$$k_a=\frac{1}{e}\cdot\{u\otimes[g(u)](k_a)-[g(u)](k_a)\otimes u\}\ , \tag{249}$$

or in a co-ordinate basis

$$k_a^{ij}=\frac{1}{e}\cdot g_{\overline{mn}}\cdot u^n\cdot(u^i\cdot k_a^{mj}-u^j\cdot k_a^{mi})\quad . \tag{250}$$

Proof: 1. Necessity. Form $h_u(k_a)h_u = 0$, it follows that

$$\begin{aligned} h_u(k_a)h_u &= 0 = g(k_a)g - \frac{1}{e} \cdot g(u) \otimes [g(u)](k_a)g - \frac{1}{e} \cdot g(k_a)[g(u)] \otimes g(u) + \\ &+ \frac{1}{e^2} \cdot [g(u)](k_a)[g(u)] \cdot g(u) \otimes g(u) \ . \end{aligned}$$

Since
$$[g(u)](k_a)[g(u)] = g_{\overline{i}\overline{m}} \cdot u^m \cdot k_a^{ij} \cdot g_{\overline{j}\overline{n}} \cdot u^n = -g_{\overline{i}\overline{m}} \cdot u^m \cdot k_a^{ij} \cdot g_{\overline{j}\overline{n}} \cdot u^n \ ,$$
we have $[g(u)](k_a)[g(u)] = 0$. Therefore,

$$g(k_a)g = \frac{1}{e} \cdot \{g(u) \otimes [g(u)](k_a)g + g(k_a)[g(u)] \otimes g(u)\} \ .$$

From the last expression and from the relation $\overline{g}[g(k_a)g]\overline{g} = k_a$, it follows that

$$\begin{aligned} k_a &= \frac{1}{e} \cdot \{u \otimes [g(u)](k_a) + (k_a)[g(u)] \otimes u\} = \\ &= \frac{1}{e} \cdot \{u \otimes [g(u)](k_a) - [g(u)](k_a) \otimes u\} \ , \end{aligned}$$

because of $(k_a)[g(u)] = -\ [g(u)](k_a)$. In a co-ordinate basis we obtain (250).

2. Sufficiency. From (249) we have

$$g(k_a)g = \frac{1}{e} \cdot \{g(u) \otimes [g(u)](k_a)g + g(k_a)[g(u)] \otimes g(u)\} \ ,$$

which is identical to $h_u(k_a)h_u = 0$.

On the other hand, after direct computations, it follows that

$$(k_a)[g(u)] = \frac{1}{2} \cdot \{(k)[g(u)] - [g(u)](k)\} \ .$$

Since $(k)[g(u)] = a$, we have the relation

$$(k_a)[g(u)] = \frac{1}{2} \cdot \{a - [g(u)](k)\} \ .$$

Then
$$k_a = \frac{1}{2 \cdot e} \cdot \{a \otimes u - u \otimes a + u \otimes [g(u)](k) - [g(u)](k) \otimes u\} \ .$$

Proposition 5 *A sufficient condition for the existence of a contravariant non-null vector field with vanishing rotation velocity* ($\omega = 0$) *is the condition*
$$k_a = 0 \ .$$

Proof: If $k_a = 0$, then it follows directly from $\omega = h_u(k_a)h_u$ that $\omega = 0$.

In a co-ordinate basis k_a is equivalent to the expression

$$u^i{}_{;l} \cdot g^{lj} - u^j{}_{;l} \cdot g^{li} = (T_{lm}{}^i \cdot g^{lj} - T_{lm}{}^j \cdot g^{li}) \cdot u^m \ .$$

On the other side, after multiplying the last expression with $g_{\overline{j}\overline{k}} \cdot u^k$ and summarizing over j, we obtain

$$a^i = u^i{}_{;k} \cdot u^k = g_{\overline{j}\overline{k}} \cdot u^k \cdot (u^j{}_{;l} - T_{lm}{}^j \cdot u^m) \cdot g^{li} = g_{\overline{j}\overline{k}} \cdot u^k \cdot k^{ji} \ ,$$

or in a form
$$a = [g(u)](k) \ .$$

Proposition 6 *The necessary condition for* $k_a = 0$ *is the condition*

$$a = [g(u)](k) \ .$$

Proof: From $k_a = 0$ and $(k_a)[g(u)] = \frac{1}{2} \cdot \{a - [g(u)](k)\}$, it follows that $a = [g(u)](k)$.

4.3 Volume-preserving (isochoric) flow

Definition 3 *A flow for which the vector field u fulfills the conditions*

$$
\begin{aligned}
{}_{\nabla}\theta &= \frac{1}{2\cdot e}\cdot(\nabla_u g)(u,u) \quad , & (251)\\
{}_{\pounds}\theta &= \frac{1}{2\cdot e}\cdot(\pounds_u g)(u,u) = \frac{1}{2\cdot e}\cdot ue \quad , & (252)
\end{aligned}
$$

is called volume-preserving (isochoric) flow.

From the last two conditions, it follows that

$$
\begin{aligned}
\theta &= {}_{\nabla}\theta - {}_{\pounds}\theta = \frac{1}{2\cdot e}\cdot[(\nabla_u g)(u,u) - ue] = -\frac{1}{e}\cdot g(u,a) \quad ,\\
\nabla_u(d\omega) &= 0 \quad , \quad \pounds_u(d\omega) = 0 \quad ,
\end{aligned}
$$

where $d\omega$ is the invariant volume element in the differentiable manifold M, considered as a model of a continuous media. Since $(\nabla_u g)(u,u) = ue - 2\cdot g(u,a)$, we have for ${}_{\nabla}\theta$ the expression

$$
{}_{\nabla}\theta = \frac{1}{2\cdot e}\cdot[ue - 2\cdot g(u,a)] \quad . \tag{253}
$$

Proposition 7 *For an inertial* $(a = \nabla_u u = 0)$ *and volume-preserving flow the following relations are fulfilled*

$$
{}_{\nabla}\theta = {}_{\pounds}\theta = \frac{1}{2\cdot e}\cdot ue \ ,
$$
$$
\theta = 0 \ .
$$

The proof is trivial. It follows from the expression for ${}_{\nabla}\theta$ and ${}_{\pounds}\theta$ in the case of a volume-preserving (isochoric) flow. From the condition $\theta = 0$, it follows that

Proposition 8 *An inertial and volume-preserving flow is an expansion-free flow* $(\theta = 0)$.

The proof follows immediately from the above relations for θ in the case of an inertial and volume-preserving flow.

Proposition 9 *For volume-preserving flow and a normalized vector field u* $[g(u,u) = const. \neq 0]$ *the following relations are valid*

$$
{}_{\nabla}\theta = -\frac{1}{e}\cdot g(u,a) \ ,
$$
$$
{}_{\pounds}\theta = 0 \ .
$$

The proof is trivial. It follows from the expression for ${}_{\nabla}\theta$ and ${}_{\pounds}\theta$ in the case of a volume-preserving (isochoric) flow.

4.4 Shear-free flow

Definition 4 *A flow with* ${}_{\nabla}\sigma = 0$ *and* ${}_{\pounds}\sigma = 0$ *is called shear-free flow.*

From the last conditions it follows that $\sigma = 0$.

The shear velocity tensor σ and the expansion velocity invariant θ are composed as the difference between the corresponding quantities induced by a transport and by its corresponding dragging along u.

A transport along u (action of ∇_u) is a motion of a material point along a line with the tangent vector u. A dragging along u (action of $\pounds_u$) is a motion of all material points lying in a vicinity (determined by the vectors $\xi_{(a)}$) of the material point at the curve with tangent vector u

$$
\begin{aligned}
\nabla_u\xi_{(a)} &= (\xi^k_{(a),i}\cdot u^i + \Gamma^k_{ji}\cdot\xi^j_{(a)}\cdot u^i)\cdot\partial_k = \xi^k_{(a);i}\cdot u^i\cdot\partial_k \quad , & (254)\\
\pounds_u\xi_{(a)} &= (\xi^k_{(a),i}\cdot u^i - u^k{}_{,i}\cdot\xi^i_{(a)})\cdot\partial_k = (\pounds_u\xi^k)\cdot\partial_k = \\
&= \nabla_u\xi_{(a)} - \nabla_{\xi_{(a)}}u - T(u,\xi_{(a)}) \quad , & (255)
\end{aligned}
$$

$$\nabla_u \xi_{(a)} - \pounds_u \xi_{(a)} = \nabla_{\xi_{(a)}} u + T(u, \xi_{(a)}) \quad .$$

The difference between a transport along u and a dragging along u of a vector field $\xi_{(a)}$ could be interpreted as a characteristic describing the (relative) change of the vector field u under the influence of the vector field $\xi_{(a)}$. Since $\nabla_{\xi_{(a)}} u$ is related to the friction velocity of the flow, the non-vanishing difference $\nabla_u \xi_{(a)} - \pounds_u \xi_{(a)}$ could characterize the friction velocity in the flow.

If we consider the structure of the relative velocity we can find the relations:

$$_{rel}v = \overline{g}[h_u(\nabla_u \xi)] = \overline{g}(h_u)(\pounds_u \xi) + \frac{l}{e} \cdot \overline{g}[h_u(a)] + \overline{g}[d(\xi)] \quad , \tag{256}$$

$$\overline{g}(h_u)(\nabla_u \xi - \pounds_u \xi) = \frac{l}{e} \cdot \overline{g}[h_u(a)] + \overline{g}[d(\xi)] \quad . \tag{257}$$

Special case: $l = g(u, \xi_{(a)}) := 0$, $\xi_{(a)} = \xi_{(a)\perp}$.

$$\begin{aligned} \overline{g}(h_u)(\nabla_u \xi_{(a)\perp} - \pounds_u \xi_{(a)\perp}) &= \overline{g}[d(\xi_{(a)\perp})] \quad , \\ (h_u)(\nabla_u \xi_{(a)\perp} - \pounds_u \xi_{(a)\perp}) &= [d(\xi_{(a)\perp})] \quad . \end{aligned} \tag{258}$$

The orthogonal to u projection of the difference $\nabla_u \xi_{(a)\perp} - \pounds_u \xi_{(a)\perp}$ is proportional to the deformation velocity tensor d. This means that d is a measure for the relative deformation induced by a transport along u and a dragging along u. This relative deformation could be an object of measurement because we can (locally) measure a deformation velocity at a point of a line with respect to the deformation velocity of its neighboring points outside the line. Usually, the deformation induced by a dragging along u is ignored by choosing the vectors u and $\xi_{(a)\perp}$ as tangent vectors to the co-ordinate lines and vice versa, by choosing the co-ordinate lines as lines with tangent vectors u and $\xi_{(a)\perp}$. Then $\pounds_u \xi_{(a)\perp} = -\pounds_{\xi_{(a)\perp}} u = 0$ and the deformation velocity tensor represent the deformation velocity for the special type of co-ordinates. this is the common (canonical) method for description of deformations in the relativistic continuous media mechanics in V_n-spaces ($n = 4$). The condition $\pounds_\xi u = 0$ has been introduced by Ehlers [5] at the beginning of all further considerations about relativistic mechanics of continuous media. If appropriate co-ordinates are imposed by the condition $\pounds_u \xi_{(a)\perp} = [u, \xi_{(a)\perp}] = 0$ a relative deformation velocity and its corresponding structures (shear, rotation, and expansion velocities) could be considered as absolute kinematic characteristics with respect to the given co-ordinates. The same is valid for the kinematic characteristics related to the deformation acceleration tensor and its corresponding structures (shear, rotation, and expansion accelerations). This is the reason for introducing and considering of many notions of continuous media mechanics under the condition $\pounds_u \xi_{(a)\perp} = 0$ or $\pounds_u \xi = 0$.

If we consider the explicit form of the shear velocity tensor (shear velocity, shear)

$$\sigma = h_u(k_s)h_u - \frac{1}{n-1} \cdot \overline{g}[h_u(k_s)h_u] \cdot h_u \tag{259}$$

we can prove the following propositions:

Proposition 10 *The necessary and sufficient condition for the existence of a non-null contravariant vector field u with vanishing shear velocity ($\sigma = 0$) is the condition*

$$\begin{aligned} k_s = \frac{1}{2 \cdot e} \cdot \{ & u \otimes a + a \otimes u + u \otimes [g(u)](k) + [g(u)](k) \otimes u - \\ & - \frac{1}{e} \cdot [ue - (\nabla_u g)(u, u)] \cdot u \otimes u \} + \\ & + \frac{1}{n-1} \cdot \theta \cdot h^u , \end{aligned} \tag{260}$$

or in a co-ordinate basis

$$\begin{aligned} h_s^{ij} = \; & \frac{1}{2 \cdot e} \cdot \{ u^i \cdot a^j + u^j \cdot a^i + u^i \cdot g_{\overline{mn}} \cdot u^n \cdot k^{mj} + u^j \cdot g_{\overline{mn}} \cdot u^n \cdot k^{mi} - \\ & - \frac{1}{e} \cdot [e_{,k} \cdot u^k - g_{km;n} \cdot u^n \cdot u^{\overline{k}} \cdot u^{\overline{m}}] \cdot u^i \cdot u^j \} + \frac{1}{n-1} \cdot \theta \cdot h^{ij} \; . \end{aligned} \tag{261}$$

Proof: 1. Necessity. From $\sigma = 0$, it follows that $h_u(k_s)h_u = \frac{1}{n-1} \cdot \overline{g}[h_u(k_s)h_u] \cdot h_u = \frac{1}{n-1} \cdot \theta \cdot h_u$. Further, from the explicit form of h_u and k_s, it follows that

$$\begin{aligned} h_u(k_s)h_u = \; & \frac{1}{n-1} \cdot \theta \cdot h_u = g(k_s)g - \frac{1}{e} \cdot \{ g(u) \otimes [g(u)](k_s)g + g(k_s)[g(u)] \otimes g(u) \} + \\ & + \frac{1}{e^2} \cdot [g(u)](k_s)[g(u)] \cdot g(u) \otimes g(u) \; , \end{aligned}$$

or

$$
\begin{aligned}
k_s &= \frac{1}{e} \cdot \{u \otimes [g(u)](k_s) + (k_s)[g(u)] \otimes u\} - \frac{1}{e^2} \cdot [g(u)](k_s)[g(u)] \cdot u \otimes u + \\
&\quad + \frac{1}{n-1} \cdot \theta \cdot \overline{g}(h_u)\overline{g} \ .
\end{aligned}
$$

Since $[g(u)](k_s) = (k_s)[g(u)]$, $(k_s)[g(u)] = \frac{1}{2} \cdot \{(k)[g(u)] + [g(u)](k)\}$, $(k)[g(u)] = a$, $(k_s)[g(u)] = \frac{1}{2} \cdot \{a + [g(u)](k)\}$, $[g(u)](k_s)[g(u)] = [g(u)](k)[g(u)] = g(u, a) = \frac{1}{2} \cdot [ue - (\nabla_u g)(u, u)]$, $\theta = \overline{g}[h_u(k_s)h_u] = \overline{g}[h_u(k)h_u]$, and $\overline{g}(h_u)\overline{g} = h^u$, the explicit form of k_s can be found as

$$
\begin{aligned}
k_s &= \frac{1}{2 \cdot e} \cdot \{u \otimes a + a \otimes u + u \otimes [g(u)](k) + [g(u)](k) \otimes u - \\
&\quad - \frac{1}{e} \cdot [ue - (\nabla_u g)(u, u)] \cdot u \otimes u\} + \\
&\quad + \frac{1}{n-1} \cdot \theta \cdot h^u \quad .
\end{aligned}
$$

2. Sufficiency. From the last expression and the above relations, it follows that $h_u(k_s)h_u = \frac{1}{n-1} \cdot \theta \cdot h_u$, and therefore $\sigma = 0$.

Proposition 11 *A sufficient condition for the existence of a non-null vector field with vanishing shear velocity ($\sigma = 0$) and expansion velocity ($\theta = 0$) is the condition*

$$
h_u(k_s)h_u = 0 \ ,
$$

identical with the condition

$$
\begin{aligned}
k_s &= \frac{1}{2 \cdot e} \cdot \{u \otimes a + a \otimes u + u \otimes [g(u)](k) + [g(u)](k) \otimes u - \\
&\quad - \frac{1}{e} \cdot [ue - (\nabla_u g)(u, u)] \cdot u \otimes u\}
\end{aligned}
$$

Proof: If $h_u(k_s)h_u = 0$, then $\theta = \overline{g}[h_u(k_s)h_u] = 0$. Therefore, $\sigma = h_u(k_s)h_u - \frac{1}{n-1} \cdot \theta \cdot h_u = 0$.
Corollary. If $h_u(k_s)h_u = 0$, then

$$
g[k_s] = \frac{1}{2 \cdot e} \cdot [ue - (\nabla_u g)(u, u)] \ . \tag{262}
$$

Proof: It follows from the above proposition that

$$
\begin{aligned}
\theta &= g[k_s] - \frac{1}{e} \cdot g(u, a) = 0 \ , \\
g[k_s] &= \frac{1}{e} \cdot g(u, a) = \frac{1}{2 \cdot e} \cdot [ue - (\nabla_u g)(u, u)] \ .
\end{aligned}
$$

4.5 Rigid flow

Definition 5 *A flow which is isochoric (volume-preserving) and shear-free is called rigid flow.*

An other definition of the notion of rigid flow could also be introduced. Let us consider the vectors $\xi_{(a)\perp}$ as infinitesimal vectors, determining a cross section in the flow orthogonal to the vector u. If the vectors $\xi_{(a)\perp}$ are Fermi-Walker transported [?] they will not change their lengths and angles between them. The cross section will move along u without any deformation. Therefore, the Fermi-Walker transport determines a motion of a cross section of a flow as a rigid body. A rigid flow is then defined as a flow with cross sections transported along a vector u by means of a Fermi-Walker transport. Since a Fermi-Walker transport is not a priori related to the kinematic characteristics of a flow, the last definition of a rigid flow is more general than the first definition using the kinematic characteristics related to the relative velocity in a flow. For a Fermi-Walker transport of type C [?] we have the relation

$$
\overline{g}[h_u(\overline{g}))({}^F\omega) - \omega](\xi_{(a)\perp}) = \frac{1}{2} \cdot h^u[(\nabla_u g)(\xi_{(a)\perp})] +
$$

$$+\overline{g}[h_u(\pounds_u\xi_{(a)\perp})]+\overline{g}[\sigma(\xi_{(a)\perp})]+\frac{1}{n-1}\cdot\theta\cdot\xi_{(a)\perp} \quad , \tag{263}$$

where ${}^F\omega$ is the covariant antisymmetric tensor of second rank from the structure of the Fermi derivative [for more details see [?], [?]]. By the use of the relations

$$\begin{aligned} h^u(\nabla_u g) &= h^u[h_u(\overline{g})(\nabla_u g)(\overline{g})h_u + \\ &+\frac{1}{e}\cdot[h_u(\overline{g})(\nabla_u g)(u)\otimes g(u)+g(u)\otimes h_u(\overline{g})(\nabla_u g)(u)]+ \\ &+\frac{1}{e^2}\cdot(\nabla_u g)(u,u)\cdot g(u)\otimes g(u)] \quad , \end{aligned} \tag{264}$$

$$\begin{aligned} h^u(h_u)\overline{g} &= h^u \quad , \\ (h_u)\overline{g}(h_u) &= h_u \quad , \end{aligned}$$

$$h^u(g)(u)=h_u[g(u)]=0 \quad ,$$

we can find a representation of the tensor ${}^F\omega$ in the form

$$\begin{aligned} {}^F\omega &= h_u(\overline{g})({}^F\omega)(\overline{g})h_u + \\ &+\frac{1}{e}\cdot[h_u(\overline{g})({}^F\omega)(u)\otimes g(u)-g(u)\otimes h_u(\overline{g})({}^F\omega)(u)] \quad . \end{aligned} \tag{265}$$

At the same time ${}^F\omega$ has the following properties:

$${}^F\omega(u) = h_u(\overline{g})({}^F\omega)(u) \quad , \tag{266}$$

$${}^F\omega(\xi_{(a)\perp}) = h_u(\overline{g})({}^F\omega)(\xi_{(a)\perp})-\frac{1}{e}\cdot(\xi_{(a)\perp})(h_u)\overline{g}({}^F\omega)(u)\cdot g(u) \quad , \tag{267}$$

where

$$\begin{aligned} (\xi_{(a)\perp})(h_u) &= g(\xi_{(a)\perp})=(\xi_{(a)\perp})g \quad , \\ (\xi_{(a)\perp})(h_u)\overline{g}({}^F\omega)(u) &= {}^F\omega(\xi_{(a)\perp},u) \quad , \\ {}^F\omega(\xi_{(a)\perp}) &= h_u(\overline{g})({}^F\omega)(\xi_{(a)\perp})-\frac{1}{e}\cdot {}^F\omega(\xi_{(a)\perp},u)\cdot g(u) \quad . \end{aligned}$$

Now ${}^F\omega$ could be represented in the form

$${}^F\omega = {}^F\omega_\perp + {}^F\widetilde{\omega} \quad , \tag{268}$$

where

$$\begin{aligned} {}^F\omega_\perp &= h_u(\overline{g})({}^F\omega)(\overline{g})h_u \quad , \\ {}^F\,\omega_\perp(u) &= -(u)({}^F\omega_\perp)=0 \quad , \end{aligned}$$

$$\begin{aligned} {}^F\widetilde{\omega} &= \frac{1}{e}\cdot[h_u(\overline{g})({}^F\omega)(u)\otimes g(u)-g(u)\otimes h_u(\overline{g})({}^F\omega)(u)] \quad , \\ {}^F\widetilde{\omega}(u) &= -(u)({}^F\widetilde{\omega})=h_u(\overline{g})({}^F\omega)(u) \quad , \\ {}^F\widetilde{\omega}(\xi_{(a)\perp}) &= -\frac{1}{e}\cdot {}^F\omega(\xi_{(a)\perp},u)\cdot g(u) \quad . \end{aligned}$$

The tensor ${}^F\omega$ contains in general terms not orthogonal to the vector u [${}^F\omega(u)=-(u)({}^F\omega)\neq 0$] in contrast to the tensor ω [$\omega(u)=-(u)(\omega)=0$]. Therefore, a rigid dynamic system is either a rigid flow or a system transported by means of a Fermi-Walker transport.

4.5.1 Rigid flow and Fermi-Walker transports

Ehlers [5] has defined a Fermi derivative ${}^e\nabla_u\xi$ in the form

$$ {}^e\nabla_u\xi_\perp := \overline{g}[h_u(\nabla_u\xi_\perp)] \quad . \tag{269}$$

If an external covariant differential operator ${}^e\nabla_u$ is chosen as [?]

$$ {}^e\nabla_u := \nabla_u - \overline{A}_u \tag{270}$$

with

$$ \overline{A}_u := \frac{1}{e}\cdot[\nabla_u u\otimes g(u) - u\otimes g(\nabla_u u)] \ ,$$

i.e. if ${}^e\nabla_u$ is chosen as

$$ {}^e\nabla_u := \nabla_u - \frac{1}{e}\cdot[\nabla_u u\otimes g(u) - u\otimes g(\nabla_u u)] \quad , \tag{271}$$

then

$$\begin{aligned} {}^e\nabla_u\xi_\perp &= \nabla_u\xi_\perp - \frac{1}{e}\cdot[\nabla_u u\otimes g(u) - u\otimes g(\nabla_u u)](\xi_\perp) = \\ &= \nabla_u\xi_\perp + \frac{1}{e}\cdot g(\nabla_u u,\xi_\perp)\cdot u \ , \end{aligned} \tag{272}$$

because of $g(u,\xi_\perp)=0$. Now, using the expression for $\nabla_u\xi_\perp$,

$$ \nabla_u\xi_\perp = \frac{1}{e}\cdot g(u,\nabla_u\xi_\perp)\cdot u + \overline{g}[h_u(\nabla_u\xi_\perp)]$$

we can find the form of the Fermi derivative, introduced by Ehlers

$$\begin{aligned} {}^e\nabla_u\xi_\perp &= \frac{1}{e}\cdot g(u,\nabla_u\xi_\perp)\cdot u + \overline{g}[h_u(\nabla_u\xi_\perp)] + \frac{1}{e}\cdot g(\nabla_u u,\xi_\perp)\cdot u = \\ &= \overline{g}[h_u(\nabla_u\xi_\perp)] - \frac{1}{e}\cdot(\nabla_u g)(u,\xi_\perp)\cdot u \ , \end{aligned} \tag{273}$$

where

$$\begin{aligned} \nabla_u[g(u,\xi_\perp)] &= u[g(u,\xi_\perp)] = 0 = \\ &= (\nabla_u g)(u,\xi_\perp) + g(\nabla_u u,\xi_\perp) + g(u,\nabla_u\xi_\perp) \quad . \end{aligned}$$

For $\nabla_u g = 0$ we have

$$ {}^e\nabla_u\xi_\perp = \overline{g}[h_u(\nabla_u\xi_\perp)] = {}_{rel}v \quad . \tag{274}$$

The last condition is not fulfilled if $\nabla_u g \neq 0$.

The notion of Fermi-Walker transport has richer contents than usually assumed on the basis of different heuristic viewpoints (Manoff 1998, 2000). In the structure of a Fermi-Walker transport a covariant antisymmetric tensor field ${}^F\omega$ of second rank plays an important role. On the other side, in the deformation velocity tensor and in the relative velocity respectively the rotation (vortex) velocity tensor ω is exactly of the type of the tensor ${}^F\omega$. This fact leads to the assumption for identification of both the tensors in (pseudo) Riemannian spaces without torsion. Such convention [5] could be unique only if we consider the kinematics of a continuous media. If we consider in addition dynamical models of substratum then there could exist other interpretations of the antisymmetric tensor ${}^F\omega$ in the structure of a Fermi-Walker transport. In $(\overline{L}_n, g)$- and (L_n, g)-spaces there is no unique relation between the rotation velocity tensor and a Fermi-Walker transport. This means that in general there is no need for a relation between the rotation velocity tensor and a Fermi-Walker transport. Only if material points in a flow are Fermi-Walker transported a relation between ${}^F\omega$ and ω could be established.

4.6 Deformation-free flow

Definition 6 *A flow with vanishing deformation velocity tensor d, i.e. with $d = 0$, is called deformation-free flow.*

If the co-ordinates in a flow are chosen in the way that $\pounds_{\xi} u = -\pounds_u \xi = 0$ then from the differential geometry in $(\overline{L}_n, g)$-spaces we have the relation

$$\nabla_u \xi - \nabla_\xi u - T(u, \xi) = 0 \ . \qquad (275)$$

The parallel transports of the deviation vector ξ along the velocity vector u and vice versa assure the vanishing of the vector of torsion $T(u, \xi)$. Therefore, the conditions

$$\begin{aligned} \pounds_u \xi &= -\pounds_\xi u = 0 \ , \\ \nabla_u \xi &= 0 \ , \\ \nabla_\xi u &= 0 \ , \end{aligned} \qquad (276)$$

should lead to deformation-free flow of a continuous media. The last two conditions ($\nabla_u \xi = 0$, $\nabla_\xi u = 0$) should by dynamically generated.. We could speak about deformations if $\nabla_u \xi \neq 0$ or $\nabla_\xi u \neq 0$, or if $\nabla_u \xi \neq 0$ and $\nabla_\xi u \neq 0$. The condition $\nabla_u \xi \neq 0$ means that the deviation vector ξ changes in the time and generates changes of the distance, the relative velocity and the relative acceleration between the material points in the media. The condition $\nabla_\xi u \neq 0$ means that the velocity vector u changes along the co-ordinate line (if $\pounds_u \xi = -\pounds_\xi u = 0$) with tangent vector ξ and this changes could be a corollary of friction between the material points in the media.

If we consider the explicit form for d

$$d := h_u(k) h_u \qquad (277)$$

we can prove the following propositions:

Proposition 12 *The necessary and sufficient condition for the existence of a non-null contravariant vector field u with vanishing deformation velocity ($d = 0$) is the condition*

$$k = \frac{1}{e} \cdot \{a \otimes u + u \otimes [g(u)](k)\} - \frac{1}{2e^2} \cdot [ue - (\nabla_u g)(u, u)] \cdot u \otimes u \ ,$$

or in a co-ordinate basis

$$\begin{aligned} k^{ij} &= \frac{1}{e} \cdot (a^i \cdot u^j + u^i \cdot k^{lj} \cdot g_{\overline{lm}} \cdot u^m) \\ &\quad - \frac{1}{2e^2} \cdot (e_{,k} \cdot u^k - g_{\overline{kl};m} \cdot u^m \cdot u^k \cdot u^l) \cdot u^i \cdot u^j \ . \end{aligned}$$

Proof: 1. Necessity. From $d = h_u(k) h_u$, after writing the explicit form of h_u, it follows that

$$\begin{aligned} d &= g(k)g - \frac{1}{e} \cdot g(u) \otimes [g(u)](k)g - \frac{1}{e} \cdot g(k)[g(u)] \otimes g(u) \\ &\quad + \frac{1}{e^2} \cdot [g(u)](k)[g(u)] \cdot u \otimes u \ . \end{aligned}$$

Since $(k)[g(u)] = a = \nabla_u u$, it follows further that

$$[g(u)](k)[g(u)] = [g(u)](a) = g(u, a) = \frac{1}{2} \cdot [ue - (\nabla_u g)(u, u)] \ .$$

Therefore,

$$\begin{aligned} d = 0 : \qquad & g(k)g = \frac{1}{e} \cdot \{g(u) \otimes [g(u)](k)g + g(a) \otimes g(u)\} \\ & - \frac{1}{e^2} \cdot g(u, a) \cdot g(u) \otimes g(u) \ . \end{aligned}$$

From $\overline{g}(g(k)g)\overline{g} = k$, we obtain

$$\begin{aligned} k &= \frac{1}{e} \cdot \{a \otimes u + u \otimes [g(u)](k)\} - \\ &\quad - \frac{1}{2 \cdot e^2} \cdot [ue - (\nabla_u g)(u, u)] \cdot u \otimes u \ . \end{aligned}$$

2. Sufficiency. From the explicit form of k, it follows that

$$\begin{aligned} g(k)g &= \frac{1}{e} \cdot g(u) \otimes [g(u)](k)g + \frac{1}{e} \cdot g(k)[g(u)] \otimes g(u) - \\ &\quad - \frac{1}{e^2} \cdot g(u, a) \cdot g(u) \otimes g(u) \end{aligned}$$

which is identical to $h_u(k)h_u = d = 0$.

Special case: $\nabla_u u = a := 0$, $\nabla_\xi g := 0$ for $\forall \xi \in T(M)$ (U_n-space), $ue = 0 : e =$const. $\neq 0$ (u is a normalized, non-null contravariant vector field).

$$d = 0 : \qquad\qquad k = \frac{1}{e} \cdot u \otimes [g(u)](k) \ ,$$

$$(k)[g(\xi)] = \frac{1}{e} \cdot u \otimes [g(u)](k)[g(\xi)] = \frac{1}{e} \cdot [g(u)](k)[g(\xi)] \cdot u \ ,$$

$$\begin{aligned}(k)[g(\xi)] &= (u^i{}_{;l} - T_{lk}{}^i \cdot u^k) \cdot g^{lm} \cdot g_{\overline{mj}} \cdot \xi^j \cdot \partial_i = \nabla_\xi u - T(\xi, u) = \\ &= \nabla_u \xi - \pounds_\xi u \ ,\end{aligned}$$

$$_{rel}v = \overline{g}[h_u(\nabla_u \xi)] = -\overline{g}(h_u)(\pounds_\xi u) \text{ for } \forall \xi \in T(M) \ . \tag{278}$$

Proposition 13 *A sufficient condition for the existence of a non-null contravariant vector field with vanishing deformation velocity ($d = 0$) is the condition*

$$k = 0 \ ,$$

equivalent to the condition

$$\nabla_\xi u = T(\xi, u) \quad \text{for } \forall \xi \in T(M) \ ,$$

or in a co-ordinate basis

$$k^{ij} = 0 : \qquad u^i{}_{;j} = T_{jk}{}^i \cdot u^k \ .$$

Proof: From $k = 0$ and $(k)[g(\xi)] = \nabla_\xi u - T(\xi, u)$ for $\forall \xi \in T(M)$, it follows that $\nabla_\xi u - T(\xi, u) = 0$ or in a co-ordinate basis $u^i{}_{;j} - T_{jk}{}^i \cdot u^k = 0$. In this case $\pounds_\xi u = \nabla_\xi u - \nabla_u \xi - T(\xi, u) = -\nabla_u \xi$.

Corollary. A deformation-free contravariant vector field u with $k = 0$ is an auto-parallel contravariant vector field.

Proof: It follows immediately from the condition $\nabla_\xi u = T(\xi, u)$ and for $\xi = u$ that $\nabla_u u = a = 0$.

Proposition 14 *The necessary condition for the existence of a deformation-free contravariant vector field u with $k = 0$ is the condition*

$$[R(u, v)]\xi = [\pounds\Gamma(u, v)]\xi \qquad \textit{for } \ \forall \xi, v \in T(M) \ ,$$

or in a co-ordinate basis

$$R^k{}_{ilj} \cdot u^l = \pounds_u \Gamma^k_{ij} \ .$$

Proof: By the use of the explicit form of the curvature operator $R(u, v)$ acting on a contravariant vector field ξ

$$[R(u, v)]\xi = \nabla_u \nabla_v \xi - \nabla_v \nabla_u \xi - \nabla_{\pounds_u v} \xi \ , \qquad \xi, \, v, \, u \in T(M) \ ,$$

and the explicit form of the deviation operator $\pounds\Gamma(u, v)$ acting on a contravariant vector field ξ

$$[\pounds\Gamma(u, v)]\xi = \pounds_u \nabla_v \xi - \nabla_v \pounds_u \xi - \nabla_{\pounds_u v} \xi$$

we obtain under the condition $\nabla_\xi u = T(\xi, u)$ (equivalent to the condition $\pounds_u \xi = \nabla_u \xi$)

$$\begin{aligned}[\pounds\Gamma(u, v)]\xi &= \pounds_u \nabla_v \xi - \nabla_v \pounds_u \xi - \nabla_{\pounds_u v} \xi = \\ &= \nabla_u \nabla_v \xi - \nabla_{\nabla_v \xi} u - T(u, \nabla_v \xi) - \nabla_v \nabla_u \xi - \nabla_{\pounds_u v} \xi = \\ &= \nabla_u \nabla_v \xi - \nabla_v \nabla_u \xi - \nabla_{\pounds_u v} \xi - [\nabla_{\nabla_v \xi} u + T(u, \nabla_v \xi)] = \\ &= [R(u, v)]\xi - [\nabla_{\nabla_v \xi} u + T(u, \nabla_v \xi)] \ .\end{aligned}$$

Since

$$\nabla_{\nabla_v \xi} u + T(u, \nabla_v \xi) = 0 \qquad \text{for} \qquad \forall v, \xi \in T(M) \ ,$$

we have

$$[R(u, v)]\xi = [\pounds\Gamma(u, v)]\xi \qquad \text{for} \quad \forall v, \xi \in T(M) \ .$$

The last condition appears as the integrability condition for the equation for u

$$\nabla_\xi u = T(\xi, u) \quad \text{for} \quad \forall \xi \in T(M) \ .$$

Proposition 15 *A deformation-free contravariant non-null vector field u with $k = 0$ is an auto-parallel non-null shear-free ($\sigma = 0$), rotation-free ($\omega = 0$) and expansion-free ($\theta = 0$) contravariant vector field with vanishing deformation acceleration ($A = 0$) [6].*

Proof: If $k = k^{ij} \cdot \partial_i \otimes \partial_j = 0$ and $\nabla_u u = a = 0$, then $k_s = k^{(ij)} \cdot \partial_i.\partial_j = \frac{1}{2} \cdot (k^{ij} + k^{ji}) \cdot \partial_i.\partial_j = 0$, and $k_a = k^{[ij]} \cdot \partial_i \wedge \partial_j = \frac{1}{2} \cdot (k^{ij} - k^{ji}) \cdot \partial_i \wedge \partial_j = 0$. Therefore, $\sigma = h_u(k_s)h_u - \frac{1}{n-1} \cdot \overline{g}[h_u(k_s)h_u] \cdot h_u = 0$, $\theta = \overline{g}[h_u(k_s)h_u] = 0$, and $\omega = h_u(k_a)h_u = 0$. From the explicit form of the deformation acceleration A (see below), it follows that $A = 0$.

From the identity for the Riemannian tensor $R^i{}_{jkl}$

$$\begin{aligned} R^i{}_{jkl} + R^i{}_{ljk} + R^i{}_{klj} \equiv\ & T_{jk}{}^i{}_{;l} + T_{lj}{}^i{}_{;k} + T_{kl}{}^i{}_{;j} + \\ & + T_{jk}{}^m \cdot T_{ml}{}^i + T_{lj}{}^m \cdot T_{mk}{}^i + T_{kl}{}^m \cdot T_{mj}{}^i \ , \end{aligned} \tag{279}$$

after contraction with g^l_i (equivalent to the action of the contraction operator $S = C$) and summation over l we obtain

$$\begin{aligned} R_{jk} - R_{kj} + R^i{}_{ijk} \equiv\ & T_{jk}{}^i{}_{;i} + T_{ij}{}^i{}_{;k} - T_{ik}{}^i{}_{;j} + \\ & + T_{jk}{}^m \cdot T_{mi}{}^i + T_{ij}{}^m \cdot T_{mk}{}^i - T_{ik}{}^m \cdot T_{mj}{}^i \ . \end{aligned} \tag{280}$$

If we introduce the abbreviations

$$_aR_{jk} := \frac{1}{2} \cdot (R_{jk} - R_{kj}) \ , \qquad T_{ji}{}^i := T_j \ , \tag{281}$$

where $T_{ik}{}^i = -T_{ki}{}^i = -T_k$, then the last expression for R_{jk} can be written in the form

$$\begin{aligned} 2 \cdot {}_aR_{jk} \equiv\ & -R^i{}_{ijk} + T_{jk}{}^i{}_{;i} + T_{ij}{}^i{}_{;k} - T_{ik}{}^i{}_{;j} + \\ & + T_{jk}{}^m \cdot T_m + T_{ij}{}^m \cdot T_{mk}{}^i - T_{ik}{}^m \cdot T_{mj}{}^i \ . \end{aligned} \tag{282}$$

Therefore, $_aR_{ij} \cdot u^j$ can be written in the form

$$\begin{aligned} 2 \cdot {}_aR_{ij} \cdot u^j + R^i{}_{ijk} \cdot u^j =\ & T_{k;j} \cdot u^j - T_{j;k} \cdot u^j + T_{jk}{}^i{}_{;i} \cdot u^j + \\ & + T_m \cdot T_{jk}{}^m \cdot u^j + T_{ij}{}^m \cdot u^j \cdot T_{mk}{}^i - T_{ik}{}^m \cdot T_{mj}{}^i \cdot u^j \ . \end{aligned} \tag{283}$$

From the other side, from $u^i{}_{;j} = T_{jl}{}^i \cdot u^l$ and $a^i = u^i{}_{;j} \cdot u^j = 0$, we have

$$u^i{}_{;j;k} = T_{jl}{}^i{}_{;k} \cdot u^l + T_{jm}{}^i \cdot T_{kl}{}^m \cdot u^l \ , \tag{284}$$

$$\begin{aligned} u^i{}_{;j;k} - u^i{}_{;k;j} =\ & -u^l \cdot R^i{}_{ljk} + T_{jk}{}^m \cdot T_{ml}{}^i \cdot u^l = \\ =\ & T_{jl}{}^i{}_{;k} \cdot u^l + T_{jm}{}^i \cdot T_{kl}{}^m \cdot u^l - \\ & - T_{kl}{}^i{}_{;j} \cdot u^l - T_{km}{}^i \cdot T_{jl}{}^m \cdot u^l \ , \end{aligned} \tag{285}$$

$$\begin{aligned} u^l \cdot R^i{}_{ljk} =\ & T_{jk}{}^m \cdot T_{ml}{}^i \cdot u^l + T_{kl}{}^i{}_{;j} \cdot u^l - T_{jl}{}^i{}_{;k} \cdot u^l + \\ & + T_{km}{}^i \cdot T_{jl}{}^m \cdot u^l - T_{jm}{}^i \cdot T_{kl}{}^m \cdot u^l \ , \end{aligned} \tag{286}$$

$$R_{lj} \cdot u^l = -T_{l;j} \cdot u^l - T_{jl}{}^i{}_{;i} \cdot u^l - T_m \cdot T_{jl}{}^m \cdot u^l \ , \tag{287}$$

$$R_{lj} \cdot u^l \cdot u^j = {}_sR_{lj} \cdot u^l \cdot u^j = I = -T_{l;j} \cdot u^l \cdot u^j = -(T_i \cdot u^i)_{;j} \cdot u^j = \dot{\theta}_1 \ . \tag{288}$$

By the use of the decompositions $R_{ij} = {}_aR_{ij} + {}_sR_{ij}$, $R_{ij} \cdot u^j = {}_aR_{ij} \cdot u^j + {}_sR_{ij} \cdot u^j$, and the above expression for $R_{lj} \cdot u^l$, we can find the following relations

$$2 \cdot {}_aR_{jk} \cdot u^j = T_{k;j} \cdot u^j - T_{j;k} \cdot u^j - 2 \cdot T_{kj}{}^i{}_{;i} \cdot u^j - 2 \cdot T_m \cdot T_{kj}{}^m \cdot u^j \ , \tag{289}$$

$$R^i{}_{ijk} \cdot u^j = (T_{kj}{}^i{}_{;i} + T_m \cdot T_{kj}{}^m + T_{ij}{}^m \cdot T_{mk}{}^i - T_{ik}{}^m \cdot T_{mj}{}^i) \cdot u^j \ , \tag{290}$$

$$2 \cdot {}_sR_{jk} \cdot u^j = -(T_{j;k} + T_{k;j}) \cdot u^j \ . \tag{291}$$

It follows that in a $(\overline{L}_n, g)$-space the projections of the symmetric part of the Ricci tensor on the non-null contravariant vector field u with $k = 0$ is depending on the covariant derivatives of T_i (respectively on the covariant derivatives of the torsion $T_{ik}{}^l$) and not on the torsion $T_{ik}{}^l$ itself.

4.7 Conformal flow

Definition 7 *A flow for which* $\pounds_u g = \lambda \cdot g$ *[or* $\pounds_u \overline{g} = -\lambda \cdot \overline{g}$*] with* $\lambda = (1/n) \cdot \overline{g}[\pounds_u g]$ *is called conformal flow.*

Proposition 16 *For a conformal flow the following relations are valid*

$$
{}_{\pounds}\theta = -\frac{n-1}{2} \cdot \lambda \quad , \qquad {}_{\pounds}\sigma = 0 \ , \quad \sigma = {}_{\nabla}\sigma ,
$$

$$
\theta = {}_{\nabla}\theta - {}_{\pounds}\theta = {}_{\nabla}\theta + \frac{n-1}{2} \cdot \lambda \quad ,
$$

$$
\pounds_u (d\omega) = \frac{n}{2} \cdot \lambda \cdot d\omega \quad .
$$

The proof follows from the explicit forms of $\pounds_u \overline{g}$, ${}_{\pounds}\theta$, and $\pounds_u(d\omega)$, $\pounds_u \overline{g} = -\overline{g}(\pounds_u g)\overline{g}$, $h_u[\overline{g}] = n-1$, under the definition of a conformal flow.

Special case: $\overline{U}_n$- and $\overline{V}_n$-spaces.

$$
{}_{\nabla}\sigma = \sigma = 0 \quad , \qquad {}_{\nabla}\theta = 0 \quad , \quad \theta = \frac{n-1}{2} \cdot \lambda \quad .
$$

4.8 Isometric flow

Definition 8 *A flow for which* $\pounds_u g = 0$ *is called isometric flow.*

Proposition 17 *For an isometric flow the following relations are valid*

$$
\begin{aligned}
{}_{\pounds}\sigma &= 0 \ , \quad {}_{\pounds}\theta = 0 \ , \qquad \pounds_u(d\omega) = 0 \quad , \\
\sigma &= {}_{\nabla}\sigma \ , \qquad \theta = {}_{\nabla}\theta \quad .
\end{aligned}
$$

5 Conclusion

In this paper the notion of relative velocity and its kinematic characteristics are introduced and considered. On an analogous basis, the notion of friction velocity and its kinematic characteristics in a continuous media are also introduced. The deformation and friction velocity tensors are found. Special types of flows show that some notions of classical continuous mechanics and hydrodynamics could be generalized without difficulties for continuous media mechanics and hydrodynamics in $(\overline{L}_n, g)$-spaces.

References

[1] Yano K., *The Theory of Lie Derivatives and its Applications* (North-Holland Pub. Co., Amsterdam, 1957).

[2] Stephani H. *Allgemeine Relativitaetstheorie.* (VEB Deutscher Verlag d. Wiss., Berlin, 1977), pp.163-166.

[3] Kramer D., Stephani H., MacCallum M., Herlt E. *Exact Solutions of Einstein's Field Equations.* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1980).

[4] Synge J.L. *Relativity: the general theory.* (North-Holland Publ. Co., Amsterdam, 1960), Ch IV., #3.

[5] Ehlers J., *Contributions to the Relativistic Mechanics of Continuous Media.* Abhandlungen d. Mainzer Akademie d. Wissenschaften, Math.-Naturwiss. Kl. Nr. **11** , 792-837 (1961). Engl. transl. Gen. Rel. and Grav. **25** (1993) 12, 1225-1266.

[6] Manoff S., Kinematics of vector fields. In Complex Structures and Vector Fields. Eds. St. Dimiev and K. Sekigawa, (World Sci.Publ., Singapore, 1995), pp. 61-113. E-print gr-qc/00 03 005.